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ABSTRACT

This twelfth unit in the SMSG secondary school mathematics series is the teacher's commentary for Unit 10. For each of the chapters in Unit 10 the goals for that chapter are discussed, the mathematics is explained, some teaching suggestions are provided, the answers to exercises are listed, and sample test questions for that chapter are suggested. (DT)

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School Mathematics Study Group

First Course in Algebra

Unit 12

First Course in Algebra

Teacher's Commentary, Part II

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Chapter 10

FACTORS AND EXPONENTS

The important ideas throughout this chapter are those of factors and factoring, and it is these ideas which we must put on a firm mathematical foundation. The traditional treatment of these ideas has tended to be one of symbol pushing with technique as its primary objective. It is our aim to make *understanding* the primary objective. Without *understanding*, a great deal of confusion is bound to occur. For example, if a student were to ask why π is not a factor of 6 (since $\pi \times \frac{6}{\pi} = 6$, and $\frac{6}{\pi}$ is a perfectly good number), or if he should ask why $x^2 + 1$ is not a factor of x $\left(\text{since } \frac{x}{x^2 + 1} \cdot (x^2 + 1) = x\right)$, he would be unable to find an answer within the framework of what is usually taught. "Don't be silly--you can see they don't come out 'even'!" is probably as satisfactory a reply as he could find.

In this chapter we shall be concerned with the algebraic *structures* which are *behind* our usual ideas of factoring. We shall be concerned mainly with the set of all positive integers which are, as we recall, closed under both addition and multiplication. It is, in fact, possible to think of the positive integers as "generated" from just "1" under addition to "obtain" any positive integer. This idea, which is rather simple for integers, will be of considerable help when we study polynomials. The basic idea is the definition of a proper factor. An integer is said to be factorable if it has a *proper* factor and is *prime* if it does not have a proper factor.

It is very important to keep in mind that these ideas of factors and factoring depend on the set over which we do the factoring. Over the set of positive integers, 4 is a factor of 12. If we permitted all integers, -4 and 4 would both be factors of 12. If we factor over the rational numbers, then $\frac{2}{3}$ is a factor of 12, as well. If we factor over the real numbers, any number is a factor of 12, and the idea has become meaningless. When we speak of factoring a positive integer, we shall always mean over the set of positive integers, unless another set is specified, since the information about factors of integers of interest to us in this chapter can be obtained from the positive integers.

The theorems we prove about factors will be stated in such a way that they immediately generalize to polynomials, for this is where we shall want them all again.

Students who have studied the SMSG 7th Grade course will have a good start in the ideas of factors, prime numbers and least common multiples. They may need to spend less time on these topics than other students would need.

10-1. Factors and Divisibility

The point of the story of the farmer with eleven cows is that the number 12 has a property which 11 does not, namely: 12 has factors other than one and itself. Specifically, 12 can be divided exactly by 2, 4, and 6. Eleven cannot be divided exactly by these numbers. The reason the stranger got his cow back was that $\frac{1}{2} + \frac{1}{4} + \frac{1}{6}$ equals $\frac{11}{12}$.

The fact that any positive integer has itself and 1 as factors follows from the property of 1: $a \cdot 1 = a$. But some positive integers have factors other than 1 and itself. We call this kind of a factor a proper factor. Observe the similarity of the factor-proper factor relation to the subset-proper subset relation. "Proper" is a word people use to indicate the really interesting cases of a particular concept. Think, for a moment, of what we mean by "proper fraction".

After the definition of proper factor, the question is asked, "Does it follow from this definition that m also can equal neither 1 nor n?" If m is a proper factor of n we mean that there is a positive integer q $(q \neq 1, q \neq n)$ such that $mq = n$. Then another name for q is $\frac{n}{m}$. If $m = 1$, then $q = \frac{n}{m} = \frac{n}{1} = n$. This is a contradiction. If $m = n$, then $q = \frac{n}{m} = \frac{n}{n} = 1$. This is a contradiction.

Problems 1-15 of Problem Set 10-1a are meant to reinforce the idea that if m is a factor of n, then $\frac{n}{m}$ is a factor of n. Students may make some decisions on the basis of divisibility rules; for example, 5 is not a factor of 24 since any number divisible by 5 would be named by a numeral which ends in 5 or 0. In problems 16 to 35 we want the student to recall his knowledge of divisibility of numbers in order to recognize proper factors of a number. These problems lead into a discussion of rules of divisibility based on the numeral representing the number.

Answers to Problem Set 10-1a; pages 249-250:

1. Yes. $2 \times 12 = 24$.
2. Yes. $3 \times 8 = 24$.
3. No. There is no integer q such that $5 \cdot q = 24$.
4. Yes. $6 \times 4 = 24$.
5. No. There is no integer q such that $9 \cdot q = 24$.
6. No. There is no integer q such that $13 \cdot q = 24$.
7. Yes. $12 \times 2 = 24$.
8. Yes. $24 \times 1 = 24$.
9. Yes. $13 \times 7 = 91$.
10. Yes. $30 \times 17 = 510$.
11. Yes. $12 \times 17 = 204$.
12. Yes. $10 \times 10{,}000 = 100{,}000$.
13. Yes. $3 \times 3367 = 10{,}101$.
14. Yes. $6 \times 3367 = 20{,}202$.
15. Yes. $12 \times 3367 = 40{,}404$.

[page 249]

The students should note that the answers for Problems 14 and 15 follow directly from Problem 13. The form of the answers may suggest how the associative property facilitates the work:

$$a \cdot b = ab$$
$$(2a)b = 2ab$$
$$(4a)b = 4ab$$

16. 85: 5; 5 × 17 = 85
17. 51: 3; 3 × 17 = 51
18. 52: 2; 2 × 26 = 52 . 4; 4 × 13 = 52
19. 29: not factorable
20. 93: 3; 3 × 31 = 93
21. 92: 2; 2 × 46 = 92 . 4; 4 × 23 = 92
22. 37: not factorable
23. 94: 2; 2 × 47 = 94
24. 55: 5; 5 × 11 = 55
25. 61: not factorable
26. 23: not factorable
27. 123: 3; 3 × 41 = 123
28. 57: 3; 3 × 19 = 57
29. 65: 5; 5 × 13 = 65
30. 122: 2; 2 × 61 = 122
31. 68: 2; 2 × 34 = 68 . 4; 4 × 17 = 68
32. 95: 5; 5 × 19 = 95
33. 129: 3; 3 × 43 = 129
34. 141: 3; 3 × 47 = 141
35. 101: not factorable

[page 250]

We would expect that students know the rules of divisibility for 2, 5, and 10, but the teacher should make certain that every student does. The following discussion is for the teacher's information. We hope that the teacher will not just lay out the rules but will aid the student to discover as much as possible for himself.

Any integer can be represented in the form $10t + u$, where t is a non-negative integer and u is an integer in the set $\{0, 1, 2, 3, \cdots, 9\}$. For example,

$$36 = 3(10) + 6,$$
$$178 = 17(10) + 8,$$
$$156237 = 15623(10) + 7.$$

The advantage of this form is that we can learn what rules of divisibility can be based on the last digit of the numeral. For example, $178 = 17(10) + 8$. Two is a factor of 10 and therefore also a factor of any multiple of 10, including 17 (10). In other words, the 17 doesn't matter since 2 is a factor of 10. Thus, whether or not 178 is divisible by 2 depends only on the last digit, 8. Since 2 is a factor of 8, 2 is a factor of 178.

Stating the previous argument more generally, we can say that, since 2 is a factor of 10, 2 is a factor of $10t$. Thus 2 is a factor of $10t + u$, if and only if 2 is a factor of u, the last digit. Notice that when $u = 0$, we use the fact that 2 is a factor of 0.

In an analogous manner, since 5 and 10 are factors of $10t$, each is a factor of $10t + u$ if and only if each is a factor of u. On the other hand, 3, 4, 7, 11, and 13 are not factors of $10t$ for every t. Thus rules for divisibility by these numbers cannot be based on only the last digit of the numeral.

We can extend this argument to numerals written in any other number base. Take the duodecimal notation, for example. Every number can be written in the form $12t + u$ where t is a non-negative integer and $0 \le u \le 11$. Each of the numbers 2, 3, 4 and 6 is a factor of $12t$; thus rules of divisibility for these numbers are based on the last digit of the numeral. That is, any number written as a base 12 numeral is divisible by 2, 3, 4, 6 or 12 if the last digit is divisible by 2, 3, 4, 6 or 12, respectively.

Returning to base 10 notation, let us examine the last two digits of any numeral. In general we could write any number $(100)h + d$ where h is a non-negative integer and d is an integer such that $0 \leq d \leq 99$. Any number which is a factor of 100 will be a factor of 100h. Therefore, any number which is a factor of 100 will be a factor of $(100)h + d$ if and only if it is a factor of d, the last two digits. Since the prime factorization of 100 is $2^2 \cdot 5^2$, any number made up of at most two 2's and two 5's will be a factor of 100. Such numbers are 2, 4, 5, 10, 20, 25, 50, and 100. But 2, 5, 10, 20, and 50 are more easily checked by a single or double application of the last digit rules, so this leaves 4 and 25. Thus, a number is divisible by 4 or 25 if and only if the number denoted by the last two digits in its decimal notation is divisible by 4 or 25.

Another interesting test is based on the sum of the digits of a numeral. If the digits of a four digit decimal numeral are a, b, c, d, the number is

$$\begin{aligned} 1000a + 100b + 10c + d &= 999a + a + 99b + b + 9c + c + d \\ &= (999a + 99b + 9c) + (a + b + c + d) \\ &= (111a + 11b + c)\, 9 + (a + b + c + d) \end{aligned}$$

A decimal numeral with any number of digits may be treated similarly.

Since 3 is a factor of 9, 3 is a factor of $(111a + 11b + c)\,9$. Hence, if 3 is a factor of $(a + b + c + d)$, it is a factor of the original number (Theorem 10-5b). Furthermore, if 3 is not a factor of $(a + b + c + d)$, then 3 is not a factor of the original number, as can be proved by assuming 3 is a factor of the original number and using Theorem 10-5d to lead to a contradiction.

We conclude, then, that divisibility by 3 can be tested by determining whether 3 is a factor of the sum of the digits of the decimal numeral.

We notice, incidentally, that 9 has a similar test for divisibility. A number is divisible by 9 if and only if the sum of the digits of its decimal numeral is divisible by 9.

When finding the prime factorization of numbers in later sections, one finds the factors 2, 3, 5, 7, •••primes•••, in that order. The student should, therefore, know divisibility rules for 2, 3, and 5, at least. The rules for 4, 9 and 25 are also helpful in other situations. There is no simple rule for divisibility by 7.

Answers to Problem Set 10-1b; page 251:

1. The numerals 28, 128, 228, 528, 3028 all have 28 as the last two digits. Four is a factor of 28 and each of the other numbers. None of the numbers 6, 106, 306, 806, or 2006 is divisible by four. None of the numbers 18, 118, or 5618 is divisible by four; but both 72 and 572 are divisible by 4. The rule is: If the number represented by the last two digits of the numeral is divisible by 4, then the number is divisible by 4.

2. The numerals 27, 207, 2007, 72, 702, and 270 are various arrangements of the numerals 2, 0, and 7. The numbers they represent are all divisible by 3. The sum of 2 and 7 is divisible by 3.

   On the other hand, 16, 106, 601, 61, and 1006 are not divisible by 3. Neither is the sum of 1 and 6 divisible by 3. The numbers 36, 306, 351, 315, and 513 are divisible by 3, and so is the sum of 3 and 6. Notice the digits 1 and 5 replace the digit 6 to suggest that it is the sum of the digits which is important. The numbers 5129 and 32122 are not divisible by 3.

   The first phrase, $(222 + 33 + 5) 9$ is divisible by 3, since 3 is a factor of 9. The second phrase, $(2+3+5+8)$, is the sum of the digits of the original number. If both of of these phrases are divisible by 3, then their sum is divisible by 3 (This is later shown in Theorem 10-5b). This is the case for the number 2358.

For any whole number the first phrase is always divisible by 3, since it always contains a factor of 9. Thus the question of the divisibility of the number rests upon the divisibility of the second phrase. Hence the rule can be stated: If the sum of the digits of the number is divisible by 3, the number itself is divisible by 3.

3. A number divisible by 9 is divisible by 3, since the presence of a factor of 9 in a number assures factors of 3. On the other hand, the presence of a factor of 3 is no assurance of a factor of 9. An example of such a number which is divisible by 3 but not by 9 is 12.

4. A number is divisible by six if it is divisible by 2 and 3. A number is divisible by 2 and 3 if the last digit of the numeral is even and the sum of the digits of the numeral is divisible by 3. Example: 156816. The sum of the digits is 27. Twenty-seven is divisible by 3. The last digit 6 is even. Therefore, 6 is a factor of 156816.

5. (a) 3 is a factor of 101,001. The test for divisibility by 3 is used.

   (b) 3 is not a factor of 37,199. The test for divisibility by 3 is used.

   (c) 6 is not a factor of 151,821. The test for divisibility by 2 is used.

   (d) 15 is a factor of 91,215. The tests for divisibility by 3 and 5 are used.

   (e) 12 is a factor of 187,326,648. The tests for divisibility by 3 and 4 are used.

10-2. Prime Numbers

We have a long range objective in the discussion of Section 10-2. Students often get the impression that $x^2 - 2$ cannot be factored when in fact a pair of its factors is $(x - \sqrt{2})(x + \sqrt{2})$. We wish, therefore, to emphasize the set of numbers over which numbers are factored. If we consider $x^2 - 2$ to be a polynomial whose variables have integer coefficients, then $x^2 - 2$ cannot be factored into polynomials of this same type. But if we consider the variables in the factors of $x^2 - 2$ to have real coefficients, then $x^2 - 2$ can be factored into polynomials of this type. For this reason we wish to spell out what kind of factors we want, not only in this chapter but in later chapters.

In this chapter we want the factors of positive integers to be positive integers. If we admit the negative integers as factors, we get the opposites of the positive factors. This adds little to our knowledge of factors or factorability.

Answers to Problem Set 10-2a; pages 252-253:

1. The "answer" is really given on page 253. The first number crossed out when crossing out all numbers having n as a common factor is n^2. Take 5 for example. The multiples of 5 are $2\cdot5$, $3\cdot5$, $4\cdot5$, $5\cdot5$, $6\cdot5$, $\cdots$.

   The multiples $2\cdot5$ and $4\cdot5$ are already crossed out because each has 2 as a proper factor. The multiple $3\cdot5$ is already crossed out because it has 3 as a proper factor. Thus the first multiple to cross out when crossing out 5's is $5\cdot5$ or 5^2 .

   To further explain why there are no numbers less than or equal to 100 to cross out when crossing out 11's, list the multiples of 11 as:
   $2\cdot11$, $3\cdot11$, $4\cdot11$, $5\cdot11$, $6\cdot11$, $7\cdot11$, $8\cdot11$, $9\cdot11$, $10\cdot11$, $11\cdot11$.
   Then go through the crossing-out process again to show that the first number to cross out would have to be $11\cdot11$.

A general argument is given in the next paragraph in case you have a student or two who would like to study it.

What we see in the sieve is that the first number which is crossed out because it is a multiple of the prime p is p^2. If a positive integer less than p^2 is not a prime, it must have a prime factor less than p; so it will have already been crossed out. Thus for example, the first number crossed out because of 11 would be 121; any number less than 121 which is not a prime must have 2, 3, 5, or 7 as a factor. To see this, let n be a positive integer which is less than p^2 and which is not a prime, (i.e., has proper factors). In symbols,

$$n < p^2$$

$$\text{and} \quad ab = n,$$

when both a and b are proper factors of n. But then

$$ab < p^2.$$

If it were true that both

$$a \geq p$$

$$\underline{\text{and}} \quad b \geq p,$$

then $ab \geq p^2$, which is a contradiction. Therefore, at least one of the proper factors a and b of n must be less than p. But then the smallest prime factor of n is also less than p, and would have been used to cross n off in the course of the sieve.

Since the set of positive integers is infinite we cannot find all the prime numbers by the Sieve of Eratosthenes. It is possible to find all the prime numbers less than some given positive integer, however, using this method. The next prime after 97 is 101.

Answers to Problem Set 10-2b; page 254:

1. The largest prime number less than 100 is 97.
   The largest prime number less than 200 is 199.
   The largest prime number less than 300 is 293.
2. The largest prime proper factor of numbers less than 100 is 47.
   The largest prime proper factor of numbers less than 200 is 97.
   The largest prime proper factor of numbers less than 300 is 149.
3. The largest number needed to cross out non-prime numbers less than 200 is 13.
   The largest number needed to cross out non-prime numbers less than 300 is 17.

10-3. Prime Factorization

We are studying the prime factorization of integers in this section. It is essential that the student learns to find the prime factors of any integer because we use this idea for reducing fractions, finding the lowest common denominator of fractions, and simplifying radicals. The Sieve of Eratosthenes provides a very natural way to obtain prime factors.

Mention of prime numbers and unique factorization is found in Studies in Mathematics, Volume III, pages 4.3 - 4.5.

We would expect Problem Set 10-3a to be worked in class.

Answers to Problem Set 10-3a; page 255:

1. 84: $2, \frac{84}{2} = 42$; $2, \frac{42}{2} = 21$; $3, \frac{21}{3} = 7$; 7 is prime.
   $84 = 2 \times 2 \times 3 \times 7$.
   16: $2, \frac{16}{2} = 8$; $2, \frac{8}{2} = 4$; $2, \frac{4}{2} = 2$.
   $16 = 2 \times 2 \times 2 \times 2$.
   37 is prime.
   $48 = 2 \times 2 \times 2 \times 2 \times 3$.
   $50 = 2 \times 5 \times 5$.
   $18 = 2 \times 3 \times 3$.

96 = 2 × 2 × 2 × 2 × 2 × 3.

99 = 3 × 3 × 11.

78 = 2 × 3 × 13.

47 is prime.

12 = 2 × 2 × 3.

2. (a) 28 (c) 77 (e) 49

(b) 30 (d) 54 (f) 36

Page 257. Exercises 10-3b are for the purpose of practicing finding the prime decompositions of integers. If the student can write these without using the method developed here so much the better. There is no particular reason why he must begin with the smallest prime and then use successively larger primes, but this procedure is systematic. The advantages of the method should be emphasized, but the teacher should not insist on its use. If students use exponents to express their answers, so much the better; but if the students do not demand exponents the teacher should avoid them at this point. The long expressions obtained become motivation for exponents in a later section.

Assuming that practically all students will use the systematic approach to prime factorization, be sure of the following:

1. The student should use rules of divisibility for at least 2, 3, and 5.
2. The student should know where he can stop.

Example:	202	2	1+0+1	is not divisible by 3
	101	101	101	is not divisible by 5
	1		$\frac{101}{7} = 14\frac{3}{7}$	

After trying to divide by 7, you are through, since 101 is less than 121, which is the first multiple of 11 not already crossed out.

Answers to Exercises 10-3b; page 257:

1. The smallest prime factor of 115 is 5, since 115 is not even and 1 + 1 + 5 is not divisible by 3.

   135: not even, 1 + 3 + 5 = 9; 3 is the smallest prime.

   321: not even, 3 + 2 + 1 = 6; 3 is the smallest prime.

   484: 2 is the smallest prime, since 2 is a factor of 4.

   539: not even, 5 + 3 + 9 not divisible by 3, last digit not 5 or 0;

   $$\frac{539}{7} = 77; \text{ hence divisible by } 7.$$

   143: 2, 3, 5, 7 fail; 11 is a factor of 143.

2. $$\begin{array}{r|l} 98 & 2 \\ 49 & 7 \\ 7 & 7 \\ 1 & \end{array} \qquad 98 = 2\times7\times7. \qquad\qquad \begin{array}{r|l} 432 & 2 \\ 216 & 2 \\ 108 & 2 \\ 54 & 2 \\ 27 & 3 \\ 9 & 3 \\ 3 & 3 \\ 1 & \end{array} \qquad 432 = 2\times2\times2\times2\times3\times3\times3.$$

   $$\begin{array}{r|l} 258 & 2 \\ 129 & 3 \\ 43 & 43 \\ 1 & \end{array}$$

   5 does not work since unit digit is not 5 or 0.

   We need not try 7 since $7^2 = 49$. 258 = 2x3x43.

   625 = 5x5x5x5.
   180 = 2x2x3x3x5.
   1024 = 2x2x2x2x2x2x2x2x2x2.
   378 = 2x3x3x3x7.
   729 = 3x3x3x3x3x3.

   Avoid the use of exponents unless the students are clamoring for them. We shall use the desire to shorten the writing of factors as motivation for exponents in a later section.

   825 = 3x5x5x11.
   576 = 2x2x2x2x2x2x3x3.
   1098 = 2x3x3x61.

   After dividing out the 3's, 5 is rejected by divisibility rule and 7 is rejected by trial. Then 61 is prime because 61 < 121, the first number for which 11 need be tried.

$3375 = 3\times3\times3\times5\times5\times5$.
$3740 = 2\times2\times5\times11\times17$.
$1311 = 3\times19\times23$. (Since 437 is in the neighborhood of $(20)^2$, we should try all primes up to and including 19.)
$5922 = 2\times3\times3\times7\times47$
$1008 = 2\times2\times2\times2\times3\times3\times7$.
$5005 = 5\times7\times11\times13$.
$444 = 2\times2\times3\times37$.
$5159 = 7\times11\times67$; 67 must be prime since we have eliminated all prime factors less than 11 and $67 < 121$.
$1455 = 3\times5\times97$; 7 fails and $97 < 121$.
$2324 = 2\times2\times7\times83$; 7 fails and $83 < 121$.

The proof of this "unique factorization" theorem is far beyond anything which the students can understand at the present time. An aid in convincing the student, perhaps, would be to begin with a number like 72, and factor it in various ways: 24×3, 9×8, $2 \times 2 \times 3 \times 2 \times 3$, 6×12, and then to factor 24×3 and 9×8 and 6×12 further until only prime factors remained. You will get the same three 2's and two 3's no matter how you do it (not always in the same order, of course, but the associative and commutative properties let you put them in any order). The fact is that 72 is "made up" of three 2's and two 3's, and this structure remains no matter how you begin and carry out your factoring of 72. If you are interested, a formal proof of the unique factorization property of the positive integers may be found in Courant and Robbins, What is Mathematics, Oxford, 1941, page 23.

10-4. Adding and Subtracting Fractions

We wish to apply the prime factorization of integers to the problem of finding the least common multiple of the denominators. We do not want blind adherence to the method developed, however, but we do want to give the student a systematic way of approaching the problem. For example, if the student were asked to add the fractions $\frac{1}{2}+\frac{1}{6}$ the least common denominator can be quickly determined by inspection, and the student should do it this way. If, however, he is asked to add $\frac{1}{57}+\frac{1}{95}$, it may not be easy to determine the least common denominator by inspection. But by prime factorization

$$57 = 3\cdot 19,$$
$$95 = 5\cdot 19,$$

and the least common denominator is $5\cdot 3\cdot 19$.

It is good technique, both here and in later work on factoring, to leave expressions in factored form as long as possible, for these factors indicate structure of the expression, which structure is otherwise forgotten. In the example on page 259, once the fractions had a common denominator, the numerators were, of course, multiplied out and combined; but, as you saw, it was to our advantage to leave the denominator in factored form until the very end. Then we know that the fraction cannot be simplified unless the numerator has as a factor one of the factors of the denominator.

Answers to Problem Set 10-4; pages 260-261:

1. (a) $\frac{2}{9}+\frac{1}{15}=\frac{2}{3\cdot 3}+\frac{1}{3\cdot 5}=\frac{2}{3\cdot 3}\cdot\frac{5}{5}+\frac{1}{3\cdot 5}\cdot\frac{3}{3}=\frac{13}{45}$

   (b) $\frac{3}{14}-\frac{4}{35}=\frac{3}{2\cdot 7}-\frac{4}{5\cdot 7}=\frac{3}{2\cdot 7}\cdot\frac{5}{5}-\frac{4}{5\cdot 7}\cdot\frac{2}{2}=\frac{7}{2\cdot 5\cdot 7}=\frac{1}{10}$

   (c) $-\frac{1}{12}+\frac{4}{26}=-\frac{1}{2\cdot 2\cdot 3}+\frac{4}{2\cdot 13}=-\frac{1}{2\cdot 2\cdot 3}\cdot\frac{13}{13}+\frac{4}{2\cdot 13}\cdot\frac{2\cdot 3}{2\cdot 3}=\frac{11}{156}$

   (d) $-\frac{5}{12}-\frac{7}{18}=-\frac{5}{2\cdot 2\cdot 3}-\frac{7}{2\cdot 3\cdot 3}=-\frac{5}{2\cdot 2\cdot 3}\cdot\frac{3}{3}-\frac{7}{2\cdot 3\cdot 3}\cdot\frac{2}{2}=-\frac{29}{36}$

   (e) $\frac{1}{85}+\frac{3}{51}=\frac{1}{5\cdot 17}+\frac{3}{3\cdot 17}=\frac{1}{5\cdot 17}\cdot\frac{3}{3}+\frac{3}{3\cdot 17}\cdot\frac{5}{5}=\frac{18}{3\cdot 5\cdot 17}=\frac{6}{85}$

   Notice that after you have added numerators, you need only check 18 for divisibility by 3, 5, and 17.

(f) $-\frac{20}{57} - \frac{7}{95} = \frac{-20}{3\cdot 19} - \frac{7}{5\cdot 19} = \frac{-100 - 21}{3\cdot 5\cdot 19} = \frac{-121}{3\cdot 5\cdot 19} = -\frac{121}{285}$.

Notice that $121 = 11^2$ and 11 is not a factor of the denominator, so the fraction cannot be reduced.

(g) $\frac{5}{21} - \frac{3}{91} = \frac{5}{3\cdot 7} - \frac{3}{7\cdot 13} = \frac{65 - 9}{3\cdot 7\cdot 13} = \frac{56}{3\cdot 7\cdot 13} = \frac{8}{39}$.

The fraction $\frac{56}{3\cdot 7\cdot 13}$ can be reduced only if 3, 7, or 13 are factors of 56.

(h) $\frac{3x}{8} + \frac{5x}{36} = \frac{3x}{2\cdot 2\cdot 2} + \frac{5x}{2\cdot 2\cdot 3\cdot 3} = \frac{27x + 10x}{2\cdot 2\cdot 2\cdot 3\cdot 3} = \frac{37x}{72}$.

(i) $\frac{1}{6} + \frac{3}{20} - \frac{2}{45} = \frac{1}{2\cdot 3} + \frac{3}{2\cdot 2\cdot 5} - \frac{2}{5\cdot 3\cdot 3} = \frac{30 + 27 - 8}{2\cdot 2\cdot 3\cdot 3\cdot 5} = \frac{49}{180}$.

(j) $\frac{3k}{10} + \frac{2k}{28} - \frac{k}{56} = \frac{3k}{2\cdot 5} + \frac{2k}{2\cdot 2\cdot 7} - \frac{k}{2\cdot 2\cdot 2\cdot 7}$

$= \frac{84k + 20k - 5k}{2\cdot 2\cdot 2\cdot 5\cdot 7} = \frac{99k}{280}$. (99 is not divisible by 2, 5, or 7.)

(k) $\frac{3a}{5} + \frac{7a}{75} - \frac{5a}{63} = \frac{3a}{5} + \frac{7a}{3\cdot 5\cdot 5} - \frac{5a}{3\cdot 3\cdot 7}$

$= \frac{945a + 147a - 125a}{3\cdot 3\cdot 5\cdot 5\cdot 7} = \frac{967a}{1575}$. (967 is not divisible by 3, 5, or 7.)

(ℓ) $\frac{805x - 6}{840}$

2. (a) $\frac{8}{15} < \frac{13}{24}$, $\frac{8}{3\cdot 5} < \frac{13}{2\cdot 2\cdot 2\cdot 3}$, $\frac{64}{2\cdot 2\cdot 2\cdot 3\cdot 5} < \frac{65}{2\cdot 2\cdot 2\cdot 3\cdot 5}$. True.

(b) $\frac{3}{16} < \frac{11}{64}$, $\frac{12}{64} < \frac{11}{64}$; False (not necessary to factor here).

(c) $\frac{14}{63} < \frac{6}{27}$, $\frac{14}{3\cdot 3\cdot 7} < \frac{6}{3\cdot 3\cdot 3}$, $\frac{2\cdot 7}{3\cdot 3\cdot 7} < \frac{2\cdot 3}{3\cdot 3\cdot 3}$, $\frac{2}{9} < \frac{2}{9}$. False.

3. (a) $\frac{1}{7}$.

$\frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \frac{1}{6}$.

$\frac{1}{7} < \frac{1}{6}$, because $7 > 6$ (If $a > b$, and $a > 0$ and $b > 0$, then $\frac{1}{a} < \frac{1}{b}$.)

(b) $$\left.\begin{aligned}\frac{4}{15} &= \frac{4}{3\cdot5} = \frac{36}{3\cdot3\cdot3\cdot5}\\ \frac{7}{27} &= \frac{7}{3\cdot3\cdot3} = \frac{35}{3\cdot3\cdot3\cdot5}\end{aligned}\right\} \quad \text{thus, } \frac{4}{15} > \frac{7}{27}\ .$$

(c) $\frac{5}{12} > \frac{5}{13}$, because $\frac{13}{5} > \frac{12}{5}$ (If $a > b$, and $a > 0$ and $b > 0$, then $\frac{1}{a} < \frac{1}{b}$.)

4. (a) $\frac{6}{27} > \frac{5}{27}$, $\frac{5}{27} > \frac{5}{28}$; thus, $\frac{6}{27} > \frac{5}{28}$. (transitive property of order)

(b) $$\left.\begin{aligned}\frac{2}{3} &= \frac{14}{3\cdot7}\\ \frac{5}{7} &= \frac{15}{3\cdot7}\end{aligned}\right\} \quad \frac{2}{3} < \frac{5}{7}\ .$$

(c) $\frac{6}{16} = \frac{2\cdot3}{2\cdot2\cdot2\cdot2} = \frac{3}{2\cdot2\cdot2}$,

$\frac{9}{24} = \frac{3\cdot3}{3\cdot2\cdot2\cdot2} = \frac{3}{2\cdot2\cdot2}$.

Thus, $\frac{6}{16} = \frac{9}{24}$.

(d) $\frac{1}{2} + \frac{1}{3} = \frac{5}{6} \cdot \frac{26}{2\cdot13} = \frac{130}{12\cdot13}$,

$\frac{11}{12} - \frac{1}{13} = \frac{143 - 12}{12\cdot13} = \frac{131}{12\cdot13}$,

Thus, $\left(\frac{1}{2} + \frac{1}{3}\right) < \left(\frac{11}{12} - \frac{1}{13}\right)$.

5. There is no algebraic approach. What we want is a number which has the largest possible proper factor. By inspection we decide $2 \times 47 = 94$ will do, because the next larger prime is 53, and 2×53 is too large.
Thus, John and Bob could ask for $94 + 47 = 141$ cents.

6. 97 plus its largest prime factor 97 is 194 cents.

*7. $\frac{1}{3}\cdot600 = 200$; therefore, $200 + \frac{1}{12}$ of anything is better.

$$200 + \frac{1}{12}(700) \quad ? \quad \frac{1}{3}(700)\ ,$$

$$200 + \frac{700}{12} \quad ? \quad 200 + \frac{100}{3}\ ,$$

$$200 + \frac{700}{12} \quad > \quad 200 + \frac{400}{12}\ ,$$

Thus, \$200 + $\frac{1}{12}$ of sales is better.

$$200 + \frac{1000}{12} \quad ? \quad \frac{1000}{3},$$

$$200 + \frac{1000}{12} \quad ? \quad 200 + \frac{400}{3},$$

$$200 + \frac{1000}{12} < 200 + \frac{1600}{12},$$

Thus, $\frac{1}{3}$ of sales is better.

If his sales amount to s dollars, then

$$200 + \frac{1}{12}s = \frac{1}{3}s$$

$$2400 + s = 4s$$

$$2400 = 3s$$

$$800 = s$$

His sales must be $800.

10-5. Some Facts About Factors

Another application of prime factorization of integers is presented in this section. Find two factors of 72 with the property that their sum is 22. This might seem like a game to the student, but we have a serious purpose. This is the kind of thinking which is done in factoring the quadratic polynomial $x^2 + 22x + 72$. Not only will the prime factorization of integers be useful later in factoring polynomials; it will provide the pattern for developing the properties of polynomials. The student will see that whatever properties he discovers for integers, analogous properties will be discovered for polynomials in Chapter 12.

There are two theorems on which we base a systematic approach to divisibility of integers. These theorems are:

10-5c. For positive integers a, b, and c, if a is a factor of b and a is not a factor of $(b+c)$, then a is not a factor of c.

10-5d. For positive integers a, b, and c, if a is a factor of b and a is a factor of $(b+c)$, then a is a factor of c.

[pages 261-263]

Theorem 10-5a is introduced to give the student the idea of proving theorems about factors. Theorem 10-5b is the generalization of Theorem 10-5a and is needed in the proof of Theorem 10-5c. The proofs of Theorems 10-5b and 10-5d are left for exercises, so their proofs are found in the Answers to Exercises 10-5, problems 7 and 8.

Let us apply Theorems 10-5c and 10-5d to the problem of finding two factors of 72 whose sum is 22. The prime factorization of 72 is $2\times2\times2\times3\times3$. If we represent the factors of 72 as b and c, we observe that b and c must have a prime factorization consisting of 2's and 3's. The question is, "Should all of the 2's go into making up b or should they be split between b and c?" Theorem 10-5d answers this question. It says, "If 2 is a factor of b and 2 is a factor of b+c, then 2 is a factor of c." So the key is the sum, 22. Since 2 is a factor of 22, the 2's must be split. Theorem 10-5c tells us that since 3 is not a factor of 22, 3 is not a factor of c (or b). That is, the 3's are not split but must go to one factor. Thus we have the following possibilities:

c	b
2×2	$2 \times 3 \times 3$
$2 \times 2 \times 3 \times 3$	2

But $2 \times 2 \times 3 \times 3$ is already larger than 22; so that possibility is eliminated. If the problem has a solution, the required factors must be 2×2 and $2 \times 3 \times 3$. These are the factors 4 and 18 and their sum is 22.

Some teachers find it helpful to use a pattern such as the following to help the students visualize what factors go where.

$$72 = 2\times2\times2\times3\times3$$
$$(\qquad) + (\qquad) = 22.$$

The factors of 72 which are before us have to be distributed in the two spaces to make a true sentence. The discussion of putting all the 3's in one space and splitting the 2's between the two spaces helps decide how to make the distribution.

[pages 261-263]

Answers to Problem Set 10-5; pages 264-266:

1. $12 = 2\times2\times3$

   $(2\times3) + (2) = 8$ 2's split for an even sum; Theorem 10-5d.

   The numbers are 6 and 2.

   $$\left.\begin{array}{l}(2\times2) + (3) = 7\\ (2\times2\times3) + (1) = 13\end{array}\right\}$$ 2's together for an odd sum; Theorem 10-5c.

   The numbers are 4 and 3, or 12 and 1.

2. $36 = 2\times2\times3\times3$

   $(3\times2\times2) + (3) = 15$

   3's split for divisibility by 3; Theorem 10-5d.

   2's together for non-divisibility by 2; Theorem 10-5c.

   The numbers are 12 and 3.

   $(2\times3\times3) + (2) = 20$

   2's split for divisibility by 2; Theorem 10-5d.

   3's together for non-divisibility by 3; Theorem 10-5c.

   The numbers are 18 and 2.

   $(2\times2) + (3\times3) = 13$

   2's together for non-divisibility by 2; Theorem 10-5c.

   3's together for non-divisibility by 3; Theorem 10-5c.

   The numbers are 4 and 9.

3. $150 = 2\times3\times5\times5$

   (a) With one factor 2 it is impossible to obtain an even sum.

   (b) $(5\times3) + (5\times2) = 25$

   $(5\times3\times2) + (5) = 35$

   The numbers are 15 and 10, or 30 and 5.

(c) $(5\times5\times2\times3) + (1) = 151$
$(5\times5\times2) + (3) = 53$
$(5\times5\times3) + (2) = 77$
$(5\times5) + (3\times2) = 31$
The numbers are 150 and 1, 50 and 3, 75 and 2, or 25 and 6.

4. $18 = 2\times3\times3$
$(3\times2) + 3 = 9$
The numbers are 6 and 3.

$(3\times3) + 2 = 11$
The numbers are 9 and 2.

5. (a) $288 = 2\times2\times2\times2\times2\times3\times3$
$(3\times3\times2) + (2\times2\times2\times2) = 18 + 16 = 34$

(b) $972 = 2\times2\times3\times3\times3\times3\times3$
$(2\times2) + (3\times3\times3\times3\times3) = 4 + 243 = 247$

(c) $216 = 2\times2\times2\times3\times3\times3$
$(2\times2\times2\times3\times3\times3) + 1 = 216 + 1 = 217$

(d) $330 = 2\times3\times5\times11$
$(3\times5) + (2\times11) = 15 + 22 = 37$

(e) $500 = 2\times2\times5\times5\times5$
$(2\times5\times5\times5) + 2 = 252$

Since this is the only possible arrangement which gives a sum which is even and not a multiple of 5 and since $252 \neq 62$, the problem has no solution.

(f) $270 = 2\times3\times3\times3\times5$
$(3\times3) + (2\times3\times5) = 9 + 30 = 39$

6. If the perimeter of a rectangle is 68 feet, then the sum of the length and width is 34 feet. The product of the length by the width is 225 square feet. Thus we want two factors of 225 whose sum is 34.
$225 = 3\times3\times5\times5.$
$(3\times3) + (5\times5) = 9 + 25 = 34.$
The length of the field is 25 feet; the width is 9 feet.

[page 264]

7. If a, b, and c are positive integers, a is a factor of b and a is a factor of c, then a is a factor of $b + c$.

Proof: $b + c = ap + aq$ where p and q are integers by definition of a factor

$= a(p + q)$ by the distributive property

Since $p + q$ is an integer, a is a factor of $b + c$.

8. For positive integers a, b, and c, if a is a factor of b and a is a factor of $(b + c)$, then a is a factor of c.

Proof: $a \cdot p = b + c$, p an integer, because a is a factor of $b + c$,

and $a \cdot q = b$, q an integer, because a is a factor of b.

Hence $a \cdot p = a \cdot q + c$

$ap + (-aq) = c$ addition property of equality

$a\left(p + (-q)\right) = c$ distributive property

Since the integers have closure with respect to addition, $\left(p + (-q)\right)$ is an integer.

Hence a is a factor of c.

9. By Theorem 10-5b since y is a factor of both $3y$ and y^2, y is a factor of $3y + y^2$. The distributive property could also be used to display y as a factor.

10. By Theorem 10-5d, if 3 is a factor of 6 and 3 is a factor of $6 + 4x$ then 3 is a factor of $4x$. 3 is a factor of $4x$ for any value of x which is a multiple of 3.

11. If x is the number of store jobs and
y is the number of house jobs, then
$50x + 150y$ is the number of cents earned.

Since 3 is a factor of $150y$ and we want 3 to be a factor of $50x + 150y$, then 3 must be a factor of $50x$ by Theorem 10-5b. This is true if x is a multiple of 3.

*11. $50x + 150y = 50(x + 3y)$

Since 2 divides 50 and since 4 must divide the expression $50(x + 3y)$, we must select even values of $x + 3y$. If x is

[pages 264-265]

even then $3y$ must be even by Theorem 10-5d. This is true when y is even. If x is odd then $3y$ must be odd, since if a sum of two integers is even, the integers must be either both even or both odd. $3y$ is odd when y is odd. Thus if the boys accept an even number of store jobs, they must accept an even number of house jobs. If they accept an odd number of store jobs, they must accept an odd number of house jobs.

12. This set of questions is intended to give more meaning to the theorems in Problem 13 by leading the student through some particular cases of them. In part (h) although 3 is a factor of 135, this fact does not "follow" from the information given.

(a)	yes	(e)	yes
(b)	no	(f)	yes
(c)	yes	(g)	yes
(d)	yes	(h)	no

13. (a) Since a is a factor of b and b is a factor of c, there exist integers n and m such that

$$an = b \quad \text{and} \quad bm = c.$$

Multiplying the members of the first equality by m, we have

$$(an)m = bm,$$

and noting the second equality, we have

$$(an)m = c.$$

By the associative property of multiplication,

$$a(nm) = c.$$

Thus, since nm is an integer by the closure property, a is a factor of c.

(b) Since a is a factor of b and a is a factor of d, there exist integers n and m such that

$$an = b \quad \text{and} \quad cm = d.$$

Thus, $(an)(cm)$ and (bd) are names for the same product:

$$(an)(cm) = bd$$

$$ac(nm) = bd, \text{ by associative and commutative properties.}$$

Thus, since nm is an integer, ac is a factor of bd.

(c) is a special case of (b)

(d) If a is a factor of b, there exists an integer n such that

$$an = b.$$

Then $(an)^2$ and b^2 are names for the same product. That is

$$a^2n^2 = b^2 .$$

Thus, since n^2 is an integer by the closure property, a^2 is a factor of b^2.

14. (a) Theorem (c)
(b) Theorem (a)
(c) Theorem (c)
(d) Theorem (c)
(e) Theorem (c)
(f) Theorem (b)

10-6. Introduction to Exponents

If your students have not already demanded a shorter way of writing 2x2x2x2x2x3x3, the exponent notation is now introduced. In the definition of

$$a^n = \underbrace{a \times a \times a \times \cdots \times a}_{n\ \text{factors}},$$

be careful to say, "a used as a factor n times", not "a multiplied by itself n times". Consider the meaning of each of these phrases: Two used as a factor 3 times means $2 \cdot 2 \cdot 2$, which is eight. Two multiplied by itself 3 times means

$2 \cdot 2 = 4$, 2 multiplied by 2 once,
$4 \cdot 2 = 8$, multiplied by 2 twice,
$8 \cdot 2 = 16$, multiplied by 2 three times, or 2^4.

Clearly, saying "multiplied by itself" can lead to confusion.

[pages 266-267]

Answers to Problem Set 10-6a; page 267:

1. 25 square inches.
   The words "squared" and "cubed" originated with the use of these operations to obtain respectively the areas of squares and volumes of cubes.
2. $64 = 2^6$; $60 = 2^2 \cdot 3 \cdot 5$; $80 = 2^4 \cdot 5$; $48 = 2^4 \cdot 3$; $128 = 2^7$; $81 = 3^4$; $49 = 7^2$; 41 is prime; $32 = 2^5$; $15 = 3 \cdot 5$; $27 = 3^3$; 29 is prime; $56 = 2^3 \cdot 7$; $96 = 2^5 \cdot 3$; $243 = 3^5$; $432 = 2^4 \cdot 3^3$; $512 = 2^9$; $576 = 2^6 \cdot 3^2$; $625 = 5^4$; $768 = 2^8 \cdot 3$; $686 = 2 \cdot 7^3$.
3. n must be a positive integer; a can be any real number.

*4. In different notation, we have used this example before, in Chapter 3, to show a non-commutative operation. It is not, in general, true that $a^b = b^a$. After the students have discovered this fact, it might be fun to ask them to discover two unequal positive integers a and b for which the latter equation is actually true. There is, in fact, only one such pair, $a = 2$ and $b = 4$, but the proof of this is beyond the present course.

The operation would be associative if $(a^b)^c$ were to equal $a^{(b^c)}$. This, again, is not true in general, as an example will show.

The one property which holds for exponents is that exponentiation is distributive over multiplication; that is, that $(ab)^c = a^c \cdot b^c$. We are not going to make any point of this terminology, particularly since we have never been careful to call the usual distributive law the distributive law of multiplication over addition. One trouble which students have with exponents is that they often assume a fictitious law which distributes exponentiation over addition; it just is not true that $(a + b)^c = a^c + b^c$. Nor is it true that $a^b a^c = a^{bc}$.

The general result

$$a^m a^n = a^{m+n}$$

can be thought of as a direct consequence of the definition of exponents. Since we decide to abbreviate

$$a \cdot a \cdot a \cdots a \text{ (p factors)}$$

to a^p, we may insert parentheses arbitrarily (by the associative property of multiplication) to write

$$\overbrace{(a \cdot a \cdots a)}^{\text{m factors}}\ \overbrace{(a \cdot a \cdots a)}^{\text{n factors}}, \text{ p factors .}$$

Then $p = m + n$, and we abbreviate this to

$$a^m a^n = a^p = a^{m+n} .$$

Answers to Problem Set 10-6b; pages 268-269:

1. (a) m^{14}
   (b) x^{12}
   (c) $2x^3$
   (d) $4x^4$
   (e) 2^4x^4
   (f) 3^7a^4
   (g) 2^9a^{10}
   (h) $x^{2a + a}$
   (i) 3^6
   (j) $3^4 \cdot 2^3$
   (k) $2^7 \cdot 3^5 \cdot 5^3$
   ( ) $2^{4a + a}$
   (m) $3^3a^3b^5$
   (n) $3^2k^2m^2t^2$
2. $8 + 27 = 35$
   $5^3 = 125$ false
3. $(2^3 \cdot 3^3) = (2 \cdot 3)^3 = 6^3$ true
4. 2 divides 6^3, 2 divides 2^3, so 2 must divide 3^3 by Theorem 10-5c. false
5. false $2^3 \cdot 3^3 = 6^3$
6. false $2^3 \cdot 3^3 = 6^3$

7. $2^3 + 2^3 = 2^3(1 + 1)$ distributive property

$= 2^3 \cdot 2$

$= 2^4$

false

8. true; by the distributive property, $2^3 + 2^3 = 2^3(1 + 1) = 2^4$.

9. $3^3 + 3^3 = 3^3(1 + 1)$ distributive property

$= 3^3 \cdot 2$

false

10. true; by the distributive property, $3^3 + 3^3 + 3^3 = 3^3(1 + 1 + 1) = 3^4$

11. false; $4^3 + 4^3 + 4^3 = 4^3(1 + 1 + 1) = 4^3 \cdot 3$.

12. true; by the distributive property, $4^3 + 4^3 + 4^3 + 4^3 = 4^3(1 + 1 + 1 + 1) = 4^4$.

Problems 7 - 12 illustrate a general result concerning powers of integers, namely $a \cdot a^{n-1} = a^n$. Your students may possibly discover this for themselves when they notice that $4^3 + 4^3 + 4^3 + 4^3 = 4 \cdot 4^3$.

13. (a) $2^3(2^2 + 2) = 2^4(2 + 1) = 2^4 \cdot 3$

(b) $x^2(2x^3 + x^2) = 2x^5 + x^4$

(c) $2x^3(2x^2 - 4x^3) = 2^2x^5 - 2^3x^6$

(d) $-3a^4(3^2a^3 - 3^3a^2) = -3^3a^7 + 3^4a^6$

(e) $(a^2 + 2a^3)(a - a^2) = (a^2 + 2a^3)a + (a^2 + 2a^3)(-a^2)$

$= a^3 + 2a^4 - a^4 - 2a^5$

$= a^3 + a^4 - 2a^5$

10-7. Further Properties of Exponents

We want to show that the rules for simplifying fractions containing powers can be generalized, but you need not emphasize these rules. The student should be encouraged to think on the basis of the definition. For example:

$$\frac{xy^3}{x^2y^2}$$

The student should reason, "There is one factor x in the numerator and two in the denominator; this is the same as just one factor x in the denominator because $\frac{x}{x} = 1$. There are three factors y in the numerator and two factors y in the denominator; this is the same as one factor y in the numerator, since $\frac{y^2}{y^2} = 1$. Avoid the use of the word "cancel", but don't treat the word as a "dirty word" should it come from a student. Just require the student to explain his use of the word in terms of the theorem, $\frac{a}{a} = 1$, $(a \neq 0)$.

Answers to Problem Set 10-7a; pages 271-272:

1. (a) 2
   (b) $\frac{1}{2}$
   (c) 1
   (d) $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$
   (e) $2^4 = 16$
   (f) $\frac{1}{2^3} = \frac{1}{8}$
   (g) $\frac{1}{2^3} = \frac{1}{8}$
   (h) $\frac{2}{3^2} = \frac{2}{9}$
2. (a) a^2 (b) $\frac{1}{m^3}$ (c) 1 (d) $\frac{1}{b^8}$
3. (a) $\frac{x^4}{4}$ (b) $3b^2$ (c) 1 (d) $\frac{4}{a}$
4. (a) $\frac{1}{a^2c^3}$ (b) $a^6b^6c^5$ (c) $a^2b^3c + a^4b^3c^4$

5. (a) $\frac{1}{5x}$ (b) $\frac{5 + x}{25x^2}$ (c) $5x$

6. (a) $\frac{9b^2}{2a^3}$ (b) $6b$ (c) $\frac{36a}{7b^6}$

7. (a) $\frac{6}{x^4y^3}$ (b) $\frac{6y^3}{17x^4a^6}$ (c) $\frac{9x^2y^3}{4a^6b^6}$

8. False. $\frac{9}{4} \neq \frac{3}{2}$

9. False. $\frac{216}{27} \neq 2$

10. True. $\frac{81}{16} = \frac{81}{16}$

11. True. $\frac{4^3 \cdot 3^3}{3^3 \cdot 4^3} = 1$

12. True. $\frac{(2 \cdot 3)^3}{3^3} = \frac{2^3 \cdot 3^3}{3^3} = 2^3$

13. The reciprocal of zero is not a number.

Page 272. We need negative exponents later in the course in order to write numerals in standard form. By studying the left side of the table, and filling in the right side of the table by formal application of "$\frac{a^m}{a^n} = a^{m-n}$", we believe the students can answer questions about a^0, a^{-1}, and a^{-2}. It should be understood that 2^0 and 2^{-2} have no meaning so far; but, if we define 2^0 to name the same number as 1, and if we define 2^{-2} to name the same number as $\frac{1}{2^2}$, then 2^0 and 2^{-2} become meaningful in terms of previously meaningful symbols. So, if we define $a^0 = 1$ and $a^{-n} = \frac{1}{a^n}$, for $a \neq 0$ and n a positive integer, then the only rule necessary for division is "$\frac{a^m}{a^n} = a^{m-n}$".

If your students find our reasons unconvincing for defining negative and zero exponents the way we did, ask them to complete the following table.

n	-3	-2	-1	0	1	2	3	4
2^n					2	4	8	16

They may get some satisfaction from observing that they can take $\frac{1}{2}$ of each number in the second row to obtain the number at its left.

Answers to Problem Set 10-7b; pages 275-277:

1. (a) 3^2
   (b) $\frac{1}{3^3}$
   (c) 1
   (d) b^2
   (e) $\frac{1}{b^3}$
   (f) $\frac{1}{10}$
   (g) $\frac{1}{10}$
   (h) 10^3
   (i) 1
   (j) $\frac{b^2}{a^3}$
   (k) $\frac{9y^3}{2x^3}$
   (ℓ) $\frac{3}{m^5}$
   (m) $\frac{1}{3t^2}$
2. (a) 1
   (b) 10^7
   (c) .007
   (d) $\frac{4}{a^3b}$
   (e) $\frac{1}{2y^4}$
   (f) $\frac{1}{8y^4}$
   (g) $\frac{3^4}{2^6}$
   (h) 1
   (i) $\frac{y^5}{2x^4}$

[pages 275-276]

3. (a) 93 millions of miles

   (b) 9.3 ten millions of miles

   (c) 9.3×10^7 is another name for 93,000,000

4. (a) $10^6 + 1$

   (b) $9a^3 + 3a$

   (c) $a^3 + 9$

   (d) $a^2 + 2 + \frac{1}{a^2}$ or $\frac{a^4 + 2a^2 + 1}{a^2}$

   (e) $a^2 - \frac{1}{a^2}$ or $\frac{a^4 - 1}{a^2}$

5. (a) 6 (c) -8 (e) 4 (g) 9

   (b) -2 (d) 14 (f) -3 (h) -2

   Problem 5 gives some experience which may be helpful to the student when he reaches Section 11-5 on square roots.

6. If n is a positive integer and $a \neq 0$, then $a^n = \frac{1}{a^{-n}}$.

   Proof:

   $a^{-n} = \frac{1}{a^n}$ by definition

   $a^n \cdot a^{-n} = a^n \cdot \frac{1}{a^n}$ multiplication property of equality

   $a^n \cdot a^{-n} = 1$ definition of reciprocal

   Thus a^{-n} is the multiplicative inverse of a^n.

*7. We have proved

   (a) $a^m \cdot a^n = a^{m+n}$ where m, n are positive integers

   (b) $\frac{a^m}{a^n} = a^{m-n}$ where m, n are positive integers

To give meaning to a^{m-n} when $m < n$ we have defined

$$a^{-n} = \frac{1}{a^n} \qquad \text{where } n \text{ is a positive integer}$$

To give meaning to a^{m-n} when $m = n$ we have defined

$$a^0 = 1$$

To prove $a^p \cdot a^q = a^{p+q}$ for all integers p and q we consider the following cases:

(1) p and q both positive

(2) either p or q positive and the other negative

(3) p and q both negative

(4) either p or q zero and the other non-zero (both zero is trivial)

Case (1) proof: Same as (a).

Case (2) proof:

Assume $p > 0$ and $q < 0$

If $q < 0$, then $-q > 0$

Consider $a^p \cdot a^q$

$$\begin{aligned}
a^p \cdot a^q &= a^p \cdot a^{-(-q)} &&, \quad a = -(-a) \\
&= a^p \cdot \frac{1}{a^{(-q)}} &&, \quad a^{-n} = \frac{1}{a^n} \\
&= \frac{a^p}{a^{-q}} &&, \quad \frac{a}{b} = a \cdot \frac{1}{b} \\
&= a^{p-(-q)} &&, \quad \frac{a^m}{a^n} = a^{m-n} \\
&= a^{p+q} &&, \quad a - b = a + (-b)
\end{aligned}$$

Case (3) proof:

If $p < 0$ and $q < 0$, then $-p > 0$ and $-q > 0$.

Consider $a^p \cdot a^q$

$$\begin{aligned} a^p \cdot a^q &= \frac{1}{a^{-p}} \cdot \frac{1}{a^{-q}} , && a^{-n} = \frac{1}{a^n} \\ &= \frac{1}{a^{-(p+q)}} , && a^m \cdot a^n = a^{m+n} \\ &= a^{p+q} , && a^{-n} = \frac{1}{a^n} \end{aligned}$$

Case (4) proof:

Assume $p \neq 0$ and $q = 0$

Consider $a^p \cdot a^q$

$$\begin{aligned} a^p \cdot a^q &= a^p \cdot a^0 && q = 0 \\ &= a^p \cdot 1 , && a^0 = 1 \\ &= a^p , && a \cdot 1 = a \\ &= a^{p+0} && a + 0 = a \\ &= a^{p+q} && q = 0 \end{aligned}$$

Once we have established that $a^p \cdot a^q = a^{p+q}$ for all integers p and q we can consider division as multiplication as shown below.

$$\frac{a^p}{a^q} = a^p \cdot a^{-q} , \quad a^{-n} = \frac{1}{a^n} \quad \text{and} \quad a^n = \frac{1}{a^{-n}}$$

Page 277:

$$\left(\frac{a}{b}\right)^3 \text{ means } \left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right) = \frac{a \cdot a \cdot a}{b \cdot b \cdot b} = \frac{a^3}{b^3} ;$$

thus $\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$.

$$\begin{aligned} (a^2b^3)^3 &= (a^2b^3)(a^2b^3)(a^2b^3) = (a^2 \cdot a^2 \cdot a^2)(b^3 \cdot b^3 \cdot b^3) \\ &= (a^{2+2+2})(b^{3+3+3}); \end{aligned}$$

thus $(a^2b^3)^3 = a^6b^9$.

[page 277]

The general result,

$$(ab)^n = a^n b^n$$

is true because $(ab)(ab) \cdots (ab)$ may be rewritten as

$$(a \cdot a \cdots a)\,(b \cdot b \cdots b)$$

by the associative and commutative properties of multiplication.

Answers to Problem Set 10-7c; Pages 277-280:

1. (a) $9a^6$ (b) $3a^6$ (c) $27a^6$ (d) $3a^9$
2. (a) $\frac{x}{3y^2}$ (b) $\frac{5x}{3y^2}$ (c) $\frac{1}{3y^2}$
3. (a) a (b) $-a$ (c) a^2 (d) $-3a^3$

   Notice the difference in meaning between $(-3)^2$ and -3^2. The first means $(-3)(-3)$, which is 9. The second is the opposite of 3^2, which is -9.
4. (a) y^2 (b) 1 (c) 16
5. (a) $-\frac{1}{z^{15}}$ (b) $\frac{1}{z^{15}}$ (c) z^{15}
6. (a) $\frac{7a^2}{20}$ (b) 1 (c) $-\frac{74a^9}{21}$
7. (a) x^a (b) x^{3a} (c) x^{6a}
8. (a) $\frac{15}{2a^2}$ (b) xy
9. (a) yes $\frac{4}{9} = \frac{4}{9}$ (d) no $\frac{5}{7} \neq \frac{5^2}{7^2}$

   (b) no $\frac{2}{3} \neq \frac{4}{9}$ (e) yes Theorem 10-5b

   (c) yes $\frac{5^2a^2}{7^2b^2} = \frac{5^2a^2}{7^2b^2}$ (f) yes Theorem 10-5b

   (g) yes Theorem 10-5b

10. (a) Suppose 3 is the number.

$$\left(2(3)\right)^2 = 36$$

$$2(3)^2 = 18$$ They are not the same.

(b) If x is the number

$$(2x)^2 = 4x^2$$

$$2(x)^2 = 2x^2$$ They are not the same.

11. (a) Area of a square of side s is s^2

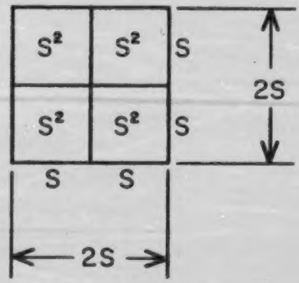
Area of a square of side $2s$ is $(2s)^2 = 4s^2$

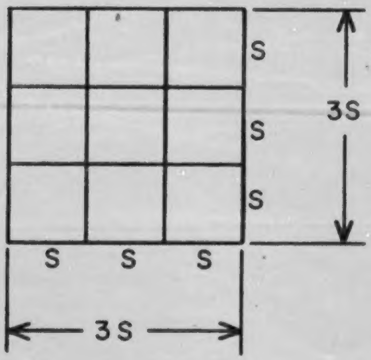
The area of the larger square is four times the area of the smaller square

(b) s^2 and $(3s)^2$

The larger square has nine times the area of the smaller square.

(c)





12. (a) $\frac{47}{24a}$

(b) $\frac{11 + x}{6x^2}$

(c) $\frac{5x - 6ay - 15a^2z}{30a^3}$

(d) $\frac{bc + ac + ab}{abc}$

(e) $\frac{55b^2 + 91ab - 245a^2}{175a^2b^2}$

13. Prove: If a^2 is odd, then a is odd.

Proof: Assume a is even. Then $a = 2q$ where q is some positive integer. Then $a^2 = 4q^2$, and a^2 is even. But this is contrary to the given hypothesis; therefore, a is not even. Therefore a is odd.

14. Prove: If a^2 is even, then a is even.

Proof: Assume a is odd. Then $a = 2n + 1$, for some positive integer n. Then $a^2 = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1$. Hence, a^2 is odd, contrary to the given hypothesis. Thus, a is not odd, and a is even.

15. Be sure to have student change fractions to lowest terms before substituting numerical values.

(a) $(-2)(2)^2(-2)^2(3)^2 = -288$

(b) $\left((-2)(2)(-2)(3)\right)^2 = 576$

(c) $\dfrac{-4a^4d}{6b^2a^3} = \dfrac{-2ad}{3b^2} = \dfrac{(-2)(2)(-3)}{3(-2)^2} = 1$

(d) $\dfrac{-3a^2}{b^2c^2} = \dfrac{-3(2)^2}{(-2)^2(3)^2} = -\dfrac{1}{3}$

(e) $\dfrac{(2)^3 + (-2)^3}{(2)^3(-2)^3} = \dfrac{8 - 8}{-2^6} = \dfrac{0}{-2^6} = 0$

(f) $\dfrac{(2 - 2 + 3)^2}{2^2 + (-2)^2 + (3)^2} = \dfrac{9}{17}$

16. (a) $(x^2 + 1)(x^3 + x^2 + 1) = x^2(x^3 + x^2 + 1) + 1\ (x^3 + x^2 + 1)$

$= x^5 + x^4 + x^2 + x^3 + x^2 + 1$

$= x^5 + x^4 + x^3 + 2x^2 + 1$

(b) $(2a^3 - b^2)(3a^2 - 2b^2) = 2a^3(3a^2 - 2b^2) - b^2(3a^2 - 2b^2)$

$= 6a^5 - 4a^3b^2 - 3a^2b^2 + 2b^4$

[page 280]

(c) $$\begin{aligned}(2x - 3y)(2x - 3y) &= 2x(2x - 3y) - 3y(2x - 3y)\\ &= 4x^2 - 6xy - 6xy + 9y^2\\ &= 4x^2 - 12xy + 9y^2\end{aligned}$$

(d) $$\begin{aligned}(a + b)^3 &= (a + b)(a + b)^2\\ &= a(a + b)^2 + b(a + b)^2\\ &= a(a^2 + 2ab + b^2) + b(a^2 + 2ab + b^2)\\ &= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3\\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

In this and the remaining chapters we are not including a chapter summary in the text. With the experience the student has now had he can make his own summary and it is much more valuable for him to do so.

Answers to Review Problems

1. (a) $\frac{5}{26}$ (d) $\frac{m - 18}{36}$

   (b) $\frac{1}{12}$ (e) $\frac{29a}{132}$

   (c) $-\frac{2}{15}$ (f) $\frac{3a^2 - 5a + 8}{12a^2}$

2. (a) $\frac{17}{10}$ (b) $\frac{7a}{9}$ (c) $\frac{3x^2}{5}$ (d) $\frac{1}{7}$

3. 130 is divisible by 2

   131 is not divisible by 2, 3, 5, 7, nor 11 so 131 is prime.

4. If x is the integer then

   $4x = 10 + 2(x + 1)$

   The integer is 6

5. (a) The solution of the equation is 93, but 93 is divisible by 3 so the truth set is empty.

   (b) $\{139\}$

(c) If there is a prime for which

$$3x^2 < 123$$

then $x^2 < 41$ where x^2 is an integer

$x < 7$ where x is an integer

the primes less than 7 are $\{2, 3, 5\}$

The left member is $3(2)^2 = 12$ when x is 2.

The left member is $3(3)^2 = 27$ when x is 3.

The left member is $3(5)^2 = 75$ when x is 5.

Thus the truth set is $\{2, 3, 5\}$.

(d) If there is a prime such that

$$|x - 10| < 3$$

then $7 < x < 13$ is an equivalent sentence.

Thus the truth set is $\{11\}$.

6. (a) $6a^3$ (c) 1 (e) $27a^3$
   (b) $\frac{y}{4x}$ (d) 2 (f) 10^4

7. (a) $a^3 + a^2$ (e) $a^3 + a^2b + ab^2 + b^3$
   (b) $x^3y^2 + xy^5$ (f) $x + x^2$
   (c) $6x^3 + 3x^2$ (g) $2 + \frac{b}{a} + \frac{a}{b}$
   (d) $mn^2 - m^2n$ (h) $x^3y^3 + xy^2 + 2x^2y^2 + 2y$

8. (a) either (g) even
   (b) either (h) odd
   (c) even (i) odd since 2 divides 2^{10} but does not divide 3^{10}.
   (d) odd
   (e) even (j) even since 2 divides both 2^{10} and 6^{10}.
   (f) odd

9. (a), (c), (e), (f), (g), (h) and (k) are non-negative

10. If x is the length of the side of the smaller square, then $x + 1$ is the length of the side of the larger square, and

$$(x + 1)^2 - x^2 = 27$$

The length of the side of the smaller square is 13 units.

11. If x is the number of nickels, $41 - x$ is the number of dimes, and

$$5x + 10(41 - x) = 335$$

Thus, Bill has 15 nickels. The information about his saving for 27 days and having more dimes than nickels is unnecessary.

12. If Sam can ride d miles into the hills, then he will ride d miles back, and

$$\frac{d}{8} + \frac{d}{12} = 5$$

The distance is 24 miles.

Chapter 10

Suggested Test Items

1. Find the prime factorization of 129, 315, 401, 3375 and 5922.
2. The next prime number after 103 is what number?
3. Why is the product of any two prime numbers which are each greater than 2, always an odd number?
4. If 5 is a factor of A and 15 is a factor of B, name a factor of $A + B$.
5. If $3^{10} + x = 123456$ is a true sentence, explain why 3 is a factor of x.
6. If $2^{10} + x = 85631$ is a true sentence, explain why 2 is not a factor of x.
7. Find two numbers whose product is 108 and whose sum is divisible by 3 but not 2.

8. Find two factors of p whose sum is s for each of the following:

	p	s
(a)	96	20
(b)	36	13
(c)	72	27
(d)	80	18
(e)	352	38
(f)	1800	85

9. Which of the following are true? Explain why or why not.

(a) $2^3 \cdot 3^3 = 6^3$

(b) $2^3 \cdot 3^2 = 6^5$

(c) $2^3 + 2^3 = 2^4$

(d) $(2^3 \cdot 2)^2 = 2^7$

(e) $2^0 + 2^0 = 2$

(f) $2^{-1} \cdot 2 = 2$

(g) $(2^2)^{-2} = 1$

(h) $2^3 + 2^2 = 2^5$

10. Simplify (explaining restrictions on the domains of the variables).

(a) $\frac{16 \cdot 3^2 x^2 y}{2^3 \cdot 27 x y^3}$

(b) $\frac{2x^3 y}{4xy}$

(c) $\frac{(2x)^3 y}{4xy}$

(d) $\frac{4}{3u} + \frac{2}{5uv} - \frac{1}{v^2}$

(e) $(x^2 + y)^2$

(f) $a^{-1} + b^{-1} + c^{-1}$

11. The squares of two successive integers differ by 11. What are the integers?

Chapter 11

RADICALS

Having prepared the way by factoring integers and studying exponents, we proceed to a study of radicals. We shall learn how to add and multiply them and how to simplify them after deciding what "simplification" means for radicals.

The method for finding approximate square roots will be the iteration method which consists of an initial estimate in standard form and its subsequent improvement. It is possible to prove, although we do not do it in the text, that this method converges in the sense that each new approximation is better than the previous, and it is also possible to show the error of each approximation.

One of the important consequences of our work on factoring of integers is that we are now able to prove that $\sqrt{2}$ is irrational. We have assumed from the beginning that $\sqrt{2}$ is a real number, in fact that $\sqrt[n]{a}$ is a real number for any positive integer n and any non-negative real number a. Our list of basic properties listed in Chapter 8 lacks one property which would be needed to prove that $\sqrt{2}$ is a real number. In this course, we assume that $\sqrt{2}$ is a real number and then prove it is not rational.

For a proof of the existence of $\sqrt{2}$ and the uniqueness of $\sqrt[n]{a}$, see Studies in Mathematics, III, pages 5.7 to 5.12.

Students who have studied the SMSG 8th Grade Course will have been exposed to some discussion about rational and irrational numbers and to a proof that $\sqrt{2}$ is irrational. They will also have worked with standard or scientific notation.

11-1. Roots

There is little confusion over the symbols $\sqrt{9}$, $\sqrt{-9}$, and $-\sqrt{9}$; but as soon as a variable appears under a square root sign we must be careful. The difficulties come from the fact that the square root symbol always indicates the positive square root and that the radicand must be non-negative. Consider $\sqrt{a^2}$. If a is positive or negative, a^2, the radicand, is positive. Thus, $\sqrt{a^2} = a$ is true if a is positive but false if a is negative. The true equation is

$$\sqrt{a^2} = |a| .$$

Examples are:

$$\sqrt{9x^2} = 3|x|$$

$$\sqrt{(-3)^2} = |-3| = 3$$

$$\sqrt{(a + b)^2} = |a + b|$$

Suppose we have a radical such as $\sqrt{16a}$. In this expression a must be non-negative; the definition of a root requires it. If a were negative we would have what is defined in a later course to be an imaginary number. Some examples are

$$\sqrt{9x^3} = \sqrt{9x^2 \cdot x} = 3|x| \sqrt{x}, \text{ but since } x \text{ must be positive,}$$
$$= 3x\sqrt{x} .$$

$$\sqrt{(x - 1)^3} = |x - 1| \sqrt{x - 1} = (x - 1)\sqrt{x - 1}, \quad x \geq 1 .$$

Page 283. If $x = 17$, then $x^2 = 289$.

If $x = .3$, then $x^2 = .09$.

If $x = -2wa^2$, then $x^2 = 4w^2a^4$.

The truth set of $x^2 = 49$ is $\{7, -7\}$.

The truth set of $x^2 = .09$ is $\{.3, -.3\}$.

The truth set of $x^2 = -4$ is $\emptyset$.

Notice that the sentence "$x^2 = n, n \geq 0$" has truth set $\{\sqrt{n}, -\sqrt{n}\}$, where the sumbol "$\sqrt{n}$" indicates the positive square root and "$-\sqrt{n}$" the negative square root. Do not allow a student to confuse $\sqrt{n}$ with the statement "find the numbers whose squares are n".

[pages 283-284]

Answers to Problem Set 11-1a; pages 284-285:

1. (a) 2
   (b) -11
   (c) $|-3| = 3$
   (d) 1.5
   (e) $\frac{7}{3}$
   (f) $.9$
   (g) 3
   (h) -1

2. (a) yes (b) no, $\sqrt{(-3)^2}$ is positive, -3 is negative.

3. (a) yes (b) yes

4. If $0 < a < b$, then exactly one of these is true: $\sqrt{a} = \sqrt{b}$, $\sqrt{a} > \sqrt{b}$, $\sqrt{a} < \sqrt{b}$, by the comparison property. If $\sqrt{a} = \sqrt{b}$, then $a = b$, contrary to hypothesis. If $\sqrt{a} > \sqrt{b}$, then $a > b$, contrary to hypothesis. Thus, the only remaining possibility is $\sqrt{a} < \sqrt{b}$.

5. No. $\sqrt{x^2}$ is positive; so $\sqrt{x^2} + 2 > 1$ for all values of x.

6. $\sqrt{(2x - 1)^2} = |2x - 1|$

   (a) $x < \frac{1}{2}$, $2x < 1$, $2x - 1 < 0$, $|2x - 1| = -(2x - 1) = 1-2x$

   (b) $x > \frac{1}{2}$, $2x > 1$, $2x - 1 > 0$, $|2x - 1| = 2x - 1$

   (c) $x = \frac{1}{2}$, $2x - 1 = 0$, $|2x - 1| = 0$

*7. The trouble with the "proof" can be best explained by inserting another step.

$$(a - b)^2 = (b - a)^2$$

$$\sqrt{(a - b)^2} = \sqrt{(b - a)^2}$$

$$|a - b| = |b - a|,$$

because $\sqrt{x^2} = |x|$. $|x| = \begin{cases} x & \text{if } x > 0, \\ -x & \text{if } x < 0. \end{cases}$

If $a - b > 0$, then $b - a < 0$.

$$a - b = -(b - a)$$
$$a - b = -b + a$$
$$a - b = a - b$$

If $a - b < 0$, then $b - a > 0$.

$$-(a - b) = b - a$$
$$-a + b = b - a$$
$$b - a = b - a$$

Page 285. If $x = 2$, then $x^3 = 8$.

If $x = -.3$, then $x^3 = -.027$.

If $x = \frac{1}{2}a^2$, then $x^4 = \frac{1}{16}a^8$.

The truth set of $x^3 = 8$ is $\{2\}$.

The truth set of $x^3 = -.072$ is $\{-.3\}$.

The truth set of $x^4 = 16$ is $\{2, -2\}$.

Help the students recognize how they can find a cube root if the number contains the same factor three times, they can find a fourth root if the number contains the same factor four times, etc.

Be sure that the students see why, in Problem 4,

$$\sqrt[3]{x^3} = x, \text{ but } \sqrt{x^2} = |x| .$$

We should not devote much time to higher roots, since we do not have fractional exponents to simplify our work. The work in the rest of this chapter is more important.

Answers to Problem Set 11-1b; page 286:

1.	(a) 3	(b) 3	(c) 3
2.	(a) 2	(b) 10	(c) 9
3.	(a) x	(b) $\lvert x\rvert$	(c) $\lvert x\rvert$
4.	(a) -5	(b) $-2y$	(c) $2y$
5.	(a) -13	(b) $\lvert -13\rvert = 13$	(c) 13^2

6. (a) $.2$ (b) $\frac{3}{10}$ (c) $.6$

7. (a) $-\frac{5}{2}$ (b) $\frac{4}{b}$, $b \neq 0$ (c) $\left|\frac{a}{b^3}\right|$, $b \neq 0$

8. (a) $4c^2$ (b) $x - 3y$ (c) 3

9. $\sqrt[4]{16} = \sqrt{4}$, $\sqrt[4]{10,000} = \sqrt{100}$, $\sqrt[4]{x^2} = \sqrt{x}$

*10. The fourth root of a number a, $\sqrt[4]{a}$, is that number which when used as a factor 4 times will yield the number a.

Similarly, $\sqrt[n]{a}$, where n is positive, is that number which when used as a factor n times will yield the number a. When a is negative and n is odd, the nth root of a will be a real number.

Example:

$$\sqrt[3]{-27} = -3.$$

11-2. Radicals

The proof that $\sqrt{2}$ is irrational which is presented in this section is interesting when we realize that here is a significant mathematical fact which can be proved at this point just on the basis of principles which have been developed in this course.

Assume there are positive integers a and b such that

$\frac{a}{b} = \sqrt{2}$, where $b \neq 0$, and a and b have no common factors.

Then $\left(\frac{a}{b}\right)^2 = 2$. Definition: If $x = \sqrt{a}$, then $x^2 = a$.

$\frac{a^2}{b^2} = 2$, since $\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$;

$a^2 = 2b^2$, by multiplying both sides by b^2;

a^2 is even. (a^2 and $2b^2$ are names for the same number, but the name $2b^2$ clearly shows a factor of 2.)

[pages 286-289]

a is even,	by theorem of Problem 14, of Problem Set 10-7c.
$a = 2c$	If a is even, then there is a positive integer c such that a is twice c.
$4c^2 = 2b^2$,	by replacing a by $2c$ in the equation $a^2 = 2b^2$ above;
$2c^2 = b^2$,	by dividing both sides by 2.
b^2 is even,	since $2c^2$ has a factor of 2, b^2 names same number.
b is even.	

Now 2 is a factor of both a and b, which contradicts our initial assumption that a and b had no common factors. Hence, $\frac{a}{b} \neq \sqrt{2}$ is true for every pair of integers a and b, $b \neq 0$.

Theorem 11-2. $\sqrt{2}$ is not a rational number.

$\sqrt{2}$ is named an irrational number. The set of rational numbers and the set of irrational numbers together make up the real numbers.

The simplest method of seeing that the square root of any positive integer which is not a perfect square, is in fact irrational, is a bit different from what we did for $\sqrt{2}$. The argument proceeds as follows. We give it for your information.

Let $\sqrt{n}$ be the number we are talking about. If n has any proper factors which are perfect squares, perform the square root operation on them (we shall be teaching this technique, and the reasons for it, in the next lesson). What is left inside the radical, then, is a product of distinct prime factors, each of them appearing just once. For example, if you were dealing with $\sqrt{45}$, you would first write it as $3\sqrt{5}$, and if we can show that $\sqrt{5}$ is irrational, then $\sqrt{45}$ certainly is also.

Thus we have a new integer, say m, inside the radical, and m is the product of prime factors each of which appears only once.

Suppose that $\sqrt{m}$ were rational, say $\frac{a}{b}$. If we multiply out, we find $a^2 = mb^2$. Now let us "count" prime factors. b^2 is a perfect square, and hence, in its prime factorization, contains every prime factor an even number of times. But then, in mb^2, the prime factors of m will appear an odd number of times, since m has every prime factor just once in its own factorization. Thus, mb^2 cannot equal a perfect square which a^2, however, is supposed to be. This is a contradiction, and thus the square root of any positive integer not a perfect square is irrational.

Now, however, we are in trouble. This proof, you see, is easier than the proof we have given in the text for the irrationality of $\sqrt{2}$. Not only is it easier, but it works in many more cases. Why then did we go through all the evens and odds argument with the students to prove the irrationality of $\sqrt{2}$?

The answer may not mean much to most of the students, but it means a great deal to mathematicians, and you should know what it is. You see, we have never proved the unique prime factorization theorem. We gave examples, we made it seem plausible, we have no hesitation in using it, but we have never proved it. Thus another proof which depends on the prime factorization is not really a proof either, it is just a convincing argument, if you like, that the result is true. A proof, like a chain, is only as strong as its weakest link. The proof in the text for the irrationality of $\sqrt{2}$ did not use prime factorization, and thus, in a sense, it is a better proof than the argument we just gave for the irrationality of square roots of non-squares.

What do we want the students to know and to use? Prime factorization is important, we have used it throughout this chapter, we will continue to use it. We want the student to use it in L.C.M. problems, in factoring problems, any other place he likes. The purpose of this discussion has been to prepare you in case the student finds this other proof of irrationality of square roots.

Many things in this course have not really been proved, and the point of the proofs in this course is not at all to make the course rigorous, but only to show the student a little bit of the nature of deductive reasoning, and to let him see that in a very real way certain facts about the real numbers are consequences of certain others. Only graduate students should be forced to go through a rigorous course on the real number system.

Answers to Problem Set 11-2; page 289:

1. In the first exercise, all we expect the student to say is that since the square of 1 is 1, 1 is too small to be the desired square root, while, since the square of 2 is 4, 2 is too large to be the desired square root. Since, however, there are no integers between 1 and 2, $\sqrt{2}$ cannot be an integer.

2. We wish to find a perfect square between $\frac{1}{3}$ and $\frac{1}{2}$. If we write equal fractions for $\frac{1}{3}$ and $\frac{1}{2}$ which have the same denominator, the numbers are more easily compared. Thus, we consider $\frac{2}{6}$ and $\frac{3}{6}$. Since a perfect square rational number would be of the form $\frac{a^2}{b^2}$, we are led to consider $\frac{12}{36}$ and $\frac{18}{36}$. Now it is easy to pick out the perfect square, $\frac{16}{36}$.

3. Assume $\frac{1}{2}\sqrt{2} = r$, where r is a rational number.

   Then $\sqrt{2} = 2r$ is an equivalent sentence.

   $2r$ is a rational number since the rational numbers are closed under multiplication. Since $\sqrt{2}$ is an irrational number by Theorem 11-2, $\sqrt{2}$ and $2r$ cannot name the same number. Thus $\sqrt{2} = 2r$ is a contradiction and our assumption that r is a rational number is false.

4. Assume $\sqrt{2} + 3 = r$, where r is a rational number.
   Then $\sqrt{2} = r + (-3)$ is an equivalent sentence.

[page 289]

Since the rational numbers close under addition, $r + (-3)$ is rational. But $\sqrt{2}$ is irrational, so $\sqrt{2} = r + (-3)$ is a contradiction. Thus, the assumption that $\sqrt{2} + 3$ was rational was false.

*5. Suppose $\sqrt{5} = \frac{p}{q}$, where p and q are integers, $q \neq 0$, and p and q have no common factors.

Then $5 = \frac{p^2}{q^2}$, by squaring both members.

$5q^2 = p^2$, by the multiplication property of equality.

Since q is an integer, q^2 is an integer.

Also, $5q^2$ and p^2 name the same number, so 5 divides p^2. Therefore 5 divides p and $p = 5n$, n an integer.

$$5q^2 = (5n)^2$$
$$q^2 = 5n^2$$

This shows 5 divides q^2 and thus 5 divides q. The statement that 5 divides p and 5 divides q is a contradiction of the assumption that p and q have no common factors.

11-3. Simplification of Radicals

Answers to Problem Set 11-3a; pages 290-291:

1. (a) $\sqrt{30}$ (b) $\sqrt{14}$ (c) 3
2. (a) $\sqrt{66}$ (b) $2\sqrt{10}$ (c) 6
3. (a) $\sqrt{2x}$, x is non-negative
   (b) $\sqrt{3yz}$, y and z are non-negative
   (c) $|x|\sqrt{3}$, x is any real number
4. (a) 0 (b) 40 (c) y^2 where y is non-negative
5. $(\sqrt{a})^2 \neq a$ for every real number a. $\sqrt{a}$ is defined only for non-negative values of a. Thus $(\sqrt{a})^2 = a$ is true for every non-negative number a.

6. If $\sqrt[3]{a}\ \sqrt[3]{b}$ is $\sqrt[3]{ab}$ we must show that $(\sqrt[3]{a}\ \sqrt[3]{b})^3$ is ab.

Proof: $\left(\sqrt[3]{a}\ \sqrt[3]{b}\right)^3 = \left(\sqrt[3]{a}\right)^3 \left(\sqrt[3]{b}\right)^3$ $\quad (ab)^n = a^n b^n$

$= ab$ definition of cube root

Thus $\sqrt[3]{ab} = \sqrt[3]{a}\ \sqrt[3]{b}$

7. (a) $\sqrt{6} + 4$

(b) $x + \sqrt{x}$, where x is non-negative

(c) $(\sqrt{a} + 1)^2 = (\sqrt{a} + 1)\sqrt{a} + (\sqrt{a} + 1)\,1$

$= a + \sqrt{a} + \sqrt{a} + 1$

$= a + 2\sqrt{a} + 1$, where a is non-negative.

(d) -1

(e) $5 + 2\sqrt{6}$

(f) $11 + \sqrt{2}$

Answers to Problem Set 11-3b; page 293:

1. (a) $2\sqrt{5}$ (b) $5\sqrt{2}$ (c) $5\sqrt{10}$ (d) $4\sqrt{5}$
2. (a) $2\sqrt{3}$ (b) $\sqrt{30}$ (c) 4 (d) $8\sqrt{3}$
3. (a) $14\sqrt{7}$ (b) $6\sqrt{13}$ (c) $5\sqrt{13}$ (d) 44
4. (a) $5\sqrt{6}$ (b) 10 (c) $9\sqrt{6}$
5. (a) $36\sqrt{7}$ (b) $2\sqrt{77}$ (c) $114\sqrt{6}$
6. (a) 5 (b) 12 (c) 7
7. (a) $2\sqrt[3]{2}$ (b) $5\sqrt[3]{2}$ (c) $\sqrt[3]{35}$ (d) 6
8. (a) $2\sqrt{14}, -2\sqrt{14}$ (b) $9\sqrt{2}, -9\sqrt{2}$ (c) $2\sqrt[3]{7}$, no negative number
9. (a) $2\sqrt{6}\,|x|$ (b) $2x\sqrt{6x}$ for all non-negative numbers x

   (c) $2x^2\sqrt{6x}$ for all non-negative numbers x

10. (a) $4a^2\sqrt{2}$ (b) $2a\sqrt[3]{4a}$ for all non-negative numbers a.

(c) $2|a|\sqrt[4]{2}$

11. (a) $\sqrt{47x}$ and x is a non-negative number

(b) $25|x|$

(c) $x^3\sqrt{5x}$ and x is a non-negative number

12. (a) $|x|\sqrt{x^2 + 1}$

(b) $|x^3|$

(c) $x^2 + |x|$

13. (a) $(2\sqrt{3x})(5\sqrt{6x})$ is defined only for non-negative numbers x

$(2\sqrt{3x})(5\sqrt{6x}) = 10\sqrt{18x^2}$ and x is non-negative

$= 30x\sqrt{2}$ and x is non-negative

(b) $\left(3\sqrt{x^2y}\right)\left(\sqrt{ay^2}\right)$ is defined only for non-negative values of y and a

$\left(3\sqrt{x^2y}\right)\left(\sqrt{ay^2}\right) = (3|x|\sqrt{y})(y\sqrt{a})$ and a and y are non-negative

$= 3|x|y\sqrt{ay}$ and a and y are non-negative numbers

(c) $1000\sqrt{3x}$ and x is a non-negative number

14. (a) 4 (b) $2\sqrt[3]{2}$ (c) 2 (d) $\sqrt[5]{16}$

15. (a) $3\sqrt[3]{a^2}$ (b) $-3b$ (c) $-3c\sqrt[3]{c}$ (d) $2\sqrt[6]{2}$

16. (a) $2x^2 = 32$

$x^2 = 16$

is equivalent to

$x = 4$ or $x = -4$.

The truth set is $\{4, -4\}$.

(b) $\{4\sqrt{3}, -4\sqrt{3}\}$

(c) $\{4, -2\}$

[page 293]

17. (a) $12\sqrt{3} - 6$ (b) $32 - 4\sqrt{2}$ (c) $-2 - 2\sqrt{6}$

Answers to Problem Set 11-4a; pages 294-295:

1. (a) $\frac{4}{5}$ (b) $\frac{1}{5}\sqrt{3}$ (c) $\frac{2}{5}\sqrt{3}$

2. (a) $\frac{1}{3}|x|$ (b) $\frac{7}{|a|}$ and $a \neq 0$ (c) $\frac{|w|}{|y|}\sqrt{3}$ and $y \neq 0$

3. (a) $\frac{|x|}{5}$ (b) $\frac{2}{3|y|}$ and $y \neq 0$ (c) $\frac{1}{3|a|}\sqrt{2}$ and $a \neq 0$

4. (a) $\frac{2}{3}$ (b) $\frac{a}{3}$ and $a > 0$

   (c) $\sqrt{\frac{y}{x}}$ and $x > 0$ and $y \geq 0$

5. (a) $\frac{1}{7}\sqrt{6}$ (b) $\frac{m}{22}\sqrt{3}$ and $m \geq 0$ (c) $\frac{1}{a}$ and $a > 0$

6. (a) $\frac{1}{5}\sqrt{15}$ (b) $\frac{\sqrt{10}}{6}$ (c) $\frac{|a|\,\sqrt{3x}}{5x}$ and $x > 0$

7. (a) $\frac{7}{5}$ (b) $\frac{7}{6}\sqrt{7}$ (c) $\frac{5}{6}\sqrt{5}$

8. To prove $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ for $a \geq 0$ and $b > 0$ we must show that $\frac{\sqrt{a}}{\sqrt{b}}$ is a square root of $\frac{a}{b}$. This will be true if $\left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2$ is $\frac{a}{b}$ by the definition of square root.

   Proof: $\left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 = \frac{(\sqrt{a})^2}{(\sqrt{b})^2}$ $\qquad \left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2}$

   $= \frac{a}{b}$ $\qquad$ definition of square root

   $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ $\qquad$ since a positive number has exactly one positive square root

Answers to Problem Set 11-4b; page 297:

1. (a) $\frac{\sqrt{3}}{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{6}\sqrt{3}$ (d) $\frac{1}{9}\sqrt{3}$

2. (a) $\frac{3}{10}\sqrt{2}$ (b) $\frac{1}{6}\sqrt{10}$ (c) $\frac{1}{2}\sqrt{7}$ (d) $\frac{1}{21}\sqrt{21}$

3. (a) $\frac{\sqrt{2}}{2}$ (b) $\frac{\sqrt{7}}{2}$ (c) $\sqrt{10}$ (d) $\frac{1}{4}\sqrt{3}$

4. (a) $\frac{1}{5}\sqrt{15b}$ and $b \geq 0$ (b) $\frac{1}{7b}\sqrt{14ab}$ and $a \geq 0$ and $b > 0$

   (c) $\frac{|x|}{3}\sqrt{2}$ (d) $\frac{1}{x^2}\sqrt{5x}$ and $x > 0$

5. (a) $\frac{1}{2}\sqrt[3]{4}$ (b) $\frac{1}{3}\sqrt[3]{3}$ (c) $\frac{1}{a}\sqrt[3]{4a}$ and $a \neq 0$

   (d) $\frac{1}{2a}\sqrt[3]{10a^2}$ and $a \neq 0$

6. (a) $\frac{1}{5}\sqrt{2}$ (b) $\frac{a^3}{15}\sqrt{5}$ and $a \geq 0$ (c) $3\sqrt{5}$

7. (a) $\frac{3\sqrt{2} - 2\sqrt{3}}{6}$ (b) 3 (c) 1

8. (a) $\frac{1}{\sqrt{x}}$ and $x > 0$ (b) $\frac{b}{\sqrt{b} + b}$ and $b > 0$

   (c) $\frac{1}{2\sqrt{7}}$

9. (a) $5 + 2\sqrt{6}$ (b) $x + 2\sqrt{x} + 1$ and $x \geq 0$

   (c) $a + \frac{1}{a} + 2$ and $a > 0$ or $\frac{a^2 + 2a + 1}{a}$

Answers to Problem Set 11-4c; page 298:

1. (a) $3\sqrt{2}$ (b) $3\sqrt{2} - 3\sqrt{3}$ (c) $19\sqrt{3}$

2. (a) $3\sqrt{2}$ (b) $\frac{14}{15}\sqrt{5}$ (c) $\sqrt{7} + 7\sqrt{3}$

3. (a) $\sqrt{34} + 2 - 2\sqrt{5}$ (b) $2\sqrt{2} + \frac{\sqrt{6}}{12}$

4. (a) $\sqrt[3]{6} + \sqrt{6}$ (b) $3\sqrt[3]{6}$ (c) $17\sqrt[4]{2}$

5. (a) $5\sqrt{a}$ (b) $a\sqrt{3a} + 2a\sqrt{a}$ (c) 0

6. (a) $[\sqrt{5}, -\sqrt{5}]$ (b) [25] (c) [5, -5]

11-5. Square Roots

There is some controversy as to whether the square root algorithm or the iteration method of approximating square roots is superior. We advocate the latter for the following reasons.

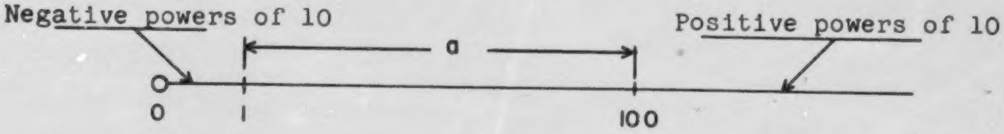
1. The iteration method can be made meaningful. It is based on the definition of the square root: If $x^2 = n$ then x is the square root of n. So the student must find a number which when squared gives n.
2. The student is more likely to realize that he is finding an approximation to $\sqrt{n}$, than when he uses the algorithm. In fact, he can be taught to estimate the size of the error.
3. The student is estimating his results; so he is not likely to make a bad error without realizing it.
4. The second approximation can very often be done mentally; always with very little arithmetic. In many cases it is all that is needed.
5. An easy division with a two digit divisor yields a result in which the error is in the fourth digit. This is sufficient for most purposes.
6. The method can be completely justified algebraically, although the justification is not given in the student text.
7. A formula for the error of any approximation can be derived.
8. The method is ideal for machine calculation.
9. If the first approximation is obtained from the slide rule, the second approximation is likely to be correct to 7 or 8 digits.

[pages 299-303]

10. The method is self-correcting. That is, if an error is made, it will still give the correct figures providing the error is not made on the last approximation.

One of the stumbling blocks that students have is the proper placement of the decimal point. We handle this problem by introducing the standard form. We write every number in the form $a \times 10^{2p}$ where $0 < a < 100$ and p is an integer ($2p$ is, therefore, an even integer). The square root of $a \times 10^{2p}$ is $\sqrt{a} \times 10^{p}$. Thus, we are always finding an approximation to a number greater than zero and less than one hundred. Aside from decimal point placement, the standard form has another advantage. Square root tables of numbers from 1 to 100 are often available in the classroom. The standard form idea brings all numbers within the scope of the table. You may want to teach square root approximations from tables if you have tables available.

The standard form idea can be referred to the positive number line for visualization. Numbers between 1 and 100 are already in the desired form or you can append a 10^{0} to them. Numbers greater than 100 are to the right of the desired interval and will be written with a positive power of 10. Positive numbers less than 1 are to the left of the desired interval and will be written with a negative power of 10. The absolute value of the exponent is equal to the number of places the decimal point is shifted.



Page 302, Examples:

(a) Since $16 < 19 < 25$, it follows that $4 < \sqrt{19} < 5$.

(b) $7 < \sqrt{54} < 8$

(c) $2 < \sqrt{5} < 3$

(d) $9 < \sqrt{96} < 10$

(e) $3 < \sqrt{11.6} < 4$

(f) $8 < \sqrt{79.42} < 9$

(g) $1 < \sqrt{1.38} < 2$

(h) $2 < \sqrt{7} < 3$

(i) $5 < \sqrt{30.2} < 6$

Answers to Problem Set 11-5a; page 303:

1. (a) 5 (b) 8 (c) 4 (d) 7 (e) 3
2. (a) 90 (b) .9 (c) .009 (d) 9
3. (a) 30 (b) 300 (c) .3 (d) 9
4. (a) 50000 (b) .0002
5. (a) 4×10^{-17} (b) 9×10^{8}

Page 303. The iteration method is very easy to explain. Suppose we consider a number a, $0 < a < 100$, and find an approximation to $\sqrt{a}$. We make a one digit estimate, x, then divide a by x, and average x and $\frac{a}{x}$. The average is the second approximation for $\sqrt{a}$. For example, find an approximation to $\sqrt{43}$. $6^2 = 36$ and $7^2 = 49$. 43 is closer to 49; so a one digit estimate of $\sqrt{43}$ is 7. Then $\frac{43}{7}$ is 6.14 and the average is $\frac{1}{2}(7 + 6.14)$, which equals 6.57. Very often the second approximation can be done mentally. For example, find $\sqrt{30}$. If the closest integer is 6, then $\frac{30}{6}$ is 5, and the average of 6 and 5 is 5.50.

Thus, we ask the student to concentrate on two ideas. First putting numbers in the form $a \times 10^{2p}$, $1 < a < 100$ and p an integer, and making nearest integer estimates of the square root of a. The nearest integer estimate will always be one digit except for values of a close to but less than 100. (For example, the nearest integer approximation of $\sqrt{97}$ is 10.) Second, dividing a by the integer estimate and averaging these two numbers. Then, for better approximations we divide and average again, each time obtaining roughly double the number of correct digits.

When we "divide and average" to get a second approximation, a natural question at this point is, "How good are the approximations we are getting?" Certainly 5 is a good estimate of the square root of 26 because 5^2 is 25 and 25 is very close to 26. If we average 5 and $\frac{26}{5}$ we get 5.100; a square root table lists $\sqrt{26}$ as 5.099020. Our average is off by only 0.001. If we divide and average again, we get $\frac{1}{2}(5.1 + \frac{26}{5.1}) = 5.0990196$. Thus, if we seek an approximation for the square root of a number whose square root is close to an integer, we expect and get good results. What, however, if we want an approximate square root of 30 ? Since $5^2 = 25$ and $6^2 = 36$ neither 5 nor 6 is a very close estimate. To answer this question we have prepared a table in which we deliberately chose the worst cases of irrational roots (the geometric mean or close to it) for which the closest integer is the first estimate. The third estimate was computed by rounding the second estimate to two digits and dividing and averaging. Thus divisions were kept quite easy.

Examination of the table shows that the second approximation is in error in the third digit and the third approximation is in error in the fourth digit. In only one case, however, is the third approximation in error by more than 0.001. In the case of $\sqrt{2}$, the error is a little less than 0.003. Remember, these are the worst cases we could have chosen.

[pages 303-306]

	First Estimate	Second Estimate (1st. Ave.)	Third Estimate (2nd. Ave.)	From Tables
$\sqrt{2}$	1	1.50	1.41667	1.414214
$\sqrt{6}$	2	2.50	2.45000	2.449490
$\sqrt{13}$	4	3.64	3.60555	3.605551
$\sqrt{21}$	5	4.60	4.58261	4.582576
$\sqrt{30}$	5	5.50	5.47727	5.477226
$\sqrt{43}$	7	6.57	6.55757	6.557439
$\sqrt{57}$	8	7.56	7.55000	7.549834
$\sqrt{73}$	9	8.56	8.54418	8.544004
$\sqrt{91}$	10	9.50	9.53947	9.539392

On the basis of the observations made on the worst cases, we set our procedure as follows.

To find the approximate square root of a number:

1. Put the number in standard form, $M \times 10^{2p}$, where $1 < M < 100$ and p is an integer.
2. Since $\sqrt{M \times 10^{2p}} = \sqrt{M} \times 10^{p}$, find the approximate square root of M.
3. As the first estimate of $\sqrt{M}$ take the closest integer, x_1.
4. Average x_1 and $\frac{M}{x}$ for the second estimate, x_2, carrying out the division $\frac{M}{x}$ to 3 digits and averaging to 3 digits.
5. Average x_2 and $\frac{M}{x_2}$ for a third estimate, x_3. Round off x_2 to two digits before dividing, in $\frac{M}{x}$, and carry the division to 4 digits and average to 4 digits. This estimate will exceed $\sqrt{M}$ by an error usually less than .002.

6. Should even more accuracy be necessary, round x_3 to 3 digits. Divide and average to 6 digits. In general, if you are sure of y digits in your divisor, you can be sure of 2y digits in your average. If great accuracy is desired, greater efficiency can be obtained by the consideration of the error at each stage. This is explained in the following discussion.

The procedure we have outlined for approximating an irrational square root seems to work. It seems to give rational numbers which are closer and closer to the irrational square root. The student may ask: Can you prove this? Let us reason as follows.

If x_1 is a positive approximation to $\sqrt{n}$ such that $x_1 > \sqrt{n}$, then

$$x_1^2 > n,$$

and

$$x_1 > \frac{n}{x_1} .$$

Then, by adding x_1 to both sides,

$$2x_1 > x_1 + \frac{n}{x_1}$$

and

$$x_1 > \frac{1}{2}\left(x_1 + \frac{n}{x_1}\right) .$$

Since the second approximation is $x_2 = \frac{1}{2}\left(x_1 + \frac{n}{x_1}\right)$, we have shown that the second approximation is always less than the first.

Let the difference between an approximation and $\sqrt{n}$ be called the error e of the approximation. Then the errors in the first two approximations are

$$e_1 = x_1 - \sqrt{n}$$

and

$$e_2 = x_2 - \sqrt{n} .$$

Thus,

$$e_2 = \frac{1}{2}\left(x_1 + \frac{n}{x_1}\right) - \sqrt{n} .$$

[pages 303-306]

By adding fractions on the right and commuting terms,

$$e_2 = \frac{x_1^2 - 2\sqrt{n}\,x_1 + n}{2x_1}$$

But the numerator on the right is a perfect square:

$$e_2 = \frac{(x_1 - \sqrt{n})^2}{2x_1} .$$

Now we observe these facts:

(1) The error e_2 in the second approximation is positive, because the square of any non-zero number is positive. Hence, x_2 is greater than $\sqrt{n}$. Then $\sqrt{n} < x_2 < x_1$, and we have shown that x_2 is closer to $\sqrt{n}$ than is x_1.

(2) The same procedure would give us the error of any approximation x in terms of the preceding approximation z,

$$\text{error of } x = \frac{(z - \sqrt{n}\,)^2}{2z} ,$$

and this error is always positive. Then x is closer to $\sqrt{n}$ than is z and $x > \sqrt{n}$. We may replace $\sqrt{n}$ by x and get the approximate formula for the error:

$$\text{error of } x \approx \frac{(z - x)^2}{2z} ,$$

where z is the preceding approximation.

To approximate $\sqrt{29}$ we find that $x_1 = 5$, $x_2 = 5.4$, and $x_3 = 5.385$; hence the error in x_3 is

$$e_3 \approx \frac{(x_2 - x_3)^2}{2x_2} ,$$

$$e_3 \approx \frac{(5.4 - 5.385)^2}{2(5.4)}$$

$$e_3 \approx \frac{(.015)^2}{10.8} \approx 0.00002.$$

[pages 303-306]

This means that x_3 is larger than $\sqrt{29}$ by about 0.00002. It shows that we could have computed x_3 to more digits. If we compute x_3 to six digits, we have

$$x_3 = \frac{1}{2}\left(5.4 + \frac{29}{5.4}\right) = \frac{1}{2}(5.4 + 5.37037) = 5.38518.$$

This is too large by about 0.00002; by subtracting the error we have

$$\sqrt{29} \approx 5.38516.$$

If the error in each approximation is taken into account, one can obtain a large number of correct digits very quickly. Let us evaluate $\sqrt{31200}$ as an example. Since $\sqrt{31200} = \sqrt{3.12} \times 10^2$, we compute $\sqrt{3.12}$.

Corrected Approx. to z	$\frac{3.12}{z}$	$x = \frac{1}{2}\left(\frac{3.12}{z} + z\right)$	Approx. Error of x
2	1.56	1.78	$\frac{(.22)^2}{2(2)} = .01$
1.78 - .01 = 1.77	1.762711	1.766355	$\frac{(0.004)^2}{3.6} = .000004$

$$\sqrt{31200} \approx 17.66355 - .000004 = 17.66351.$$

You might wonder how far to carry out the division $\frac{3.12}{1.77}$. The error of x is given by

$$e_3 \approx \frac{(z - x)^2}{2z}$$

$$z - x = z - \frac{1}{2}\left(\frac{n}{z} + z\right)$$

$$= z - \frac{n}{2z} - \frac{z}{2}.$$

$$= \frac{z}{2} - \frac{n}{2z}.$$

$$z - x = \frac{1}{2}\left(z - \frac{n}{z}\right).$$

[pages 303-306]

In this example $z = 1.77$ and $\frac{n}{z} = \frac{3.12}{1.77} = 1.762 \cdots$. When $\frac{n}{z}$ is carried out far enough so its digits begin to differ from the digits of z, we can find $z - x$. In this case

$$z - x = \frac{1}{2}\left(z - \frac{n}{z}\right) = \frac{1}{2}(1.77 - 1.762) = .004.$$

Now we can find e_3 approximately;

$$e_3 \approx \frac{(.004)^2}{2(1.8)} = .000004.$$

Thus we know that if we continue the division, $\frac{n}{z}$, to six decimal places and average, the error will occur in the sixth decimal place.

Answers to Problem Set 11-5b; pages 306-308:

1. (a) $\sqrt{796} = \sqrt{7.96} \times 10^1 \qquad p = 3 \quad q = \frac{7.96}{3}$

$$\frac{p+q}{2} = \frac{1}{2}\left(3 + \frac{7.96}{3}\right)$$

$$\frac{p+q}{2} \approx 2.82$$

$$\sqrt{796} \approx 28.2$$

(b) $\sqrt{73} \qquad p = 9 \quad q = \frac{73}{9}$

$$\frac{p+q}{2} = \frac{1}{2}\left(9 + \frac{73}{9}\right)$$

$$\frac{p+q}{2} \approx 8.56$$

$$\sqrt{73} \approx 8.56$$

(c) 2.97

(d) 554

(e) 0.0763

(f) 3170

2. (a) $\sqrt{0.00470} = \sqrt{47} \times 10^{-2}$

p	$q = \frac{47}{p}$	$\frac{p+q}{2}$
7	6.71	6.86
6.9	6.812	6.856

$\sqrt{0.00470} \approx 0.06856$

(b) $\sqrt{0.273} = \sqrt{27.3} \times 10^{-1}$

p	$q = \frac{27.3}{p}$	$\frac{p+q}{2}$
5	5.46	5.23
5.2	5.250	5.225

$\sqrt{0.273} \approx .5225$

(c) 72.66

(d) 1.772 Note: The thousandths digit could be either 2 or 3. Since the average is, after the first approximation, always high; it is better to "round down".

(e) 265.1

(f) 708.5

3. (a) $\sqrt{0.0072} = \sqrt{72} \times \sqrt{10^{-4}}$

$\approx 8.485 \times 10^{-2}$

$\approx .08485$

(b) $\sqrt{720000} \approx 848.5$

(c) $\sqrt{.72} \approx .8485$

(d) $\sqrt{0.08} \approx .2828$

(e) $\sqrt{800} \approx 28.28$

(f) $\sqrt{8,000,000} \approx 2828$

You may wish to do more than this with square root tables. We did not, however, include a square root table in the text because some teachers prefer not to use one at this point.

4. (a) $x^2 = 0.0124$

$x = \sqrt{0.0124}$ or $x = -\sqrt{0.0124}$

The truth set is $\{\sqrt{0.0124}, -\sqrt{0.0124}\}$.

$\sqrt{0.0124} \approx 0.112$

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Thus approximations for the elements of the truth set are 0.112 and -0.112.

(b) 22.9 and -22.9

5. 361 feet

6. 12 feet, The nearest foot.

7. 8.45 cm

Answers to Review Exercises; pages 308-311:

1. (a) $2\sqrt{3}$

(b) $\frac{1}{6}$

(c) $2\sqrt{2a}$ and $a \geq 0$

(d) $\frac{4}{3}\sqrt{3}$

(e) $3|x|\sqrt{2}$

(f) 12

(g) $-\sqrt{2}$

(h) $a^3b^3c^2$

(i) $2 + 2\sqrt{3}$

2. (a) $\sqrt{3}$

(b) 10

(c) $2|a + b|$

(d) $3^a \cdot 2^b$ and a and b are integers

(e) $-\frac{1}{4}\sqrt{2}$

(f) $2 - 2\sqrt{3}$

(g) $3\sqrt[3]{2}$

(h) $-4x\sqrt{2xy}$ and x and y are both non-negative

(i) $\frac{1}{2}\sqrt{6} + 1$

3. (a) $2|a|\sqrt{3}$

(b) $\frac{\sqrt{6} + 2}{2}$

(c) 1

(d) $\frac{2|m|\sqrt{q}}{q} + 7|m|q\sqrt{2q}$ and $q > 0$

(e) $\frac{1}{6}\sqrt{30}$

(f) $\frac{\sqrt[3]{4x^2}}{10x}$ and $x \neq 0$

(g) $3p^2\sqrt{2}$ and $p \geq 0$

(h) $\frac{1}{2a}\sqrt[3]{2a^2} - 2a\sqrt[3]{2a^2} = \sqrt[3]{2a}\left(\frac{1 - 4a^2}{2a}\right)$ where $a \neq 0$

(i) $2\sqrt{a^2 + b^2}$

4. (a) $\{4\}$. Notice that $\sqrt{x} = 2$ is not a sentence if $x < 0$. Squaring both members of $\sqrt{x} = 2$ gives the equivalent sentence $x = 4$ but squaring both members of an equation does not always give an equivalent sentence. This fact will be studied in Chapter 13.

(b) $\{64\}$

(c) $\{\sqrt{2}, -\sqrt{2}\}$.

(d) all m such that $-4 \leq m \leq 4$

(e) $\{8\}$

(f) $2|x| + \sqrt{x^2} = 3$

is equivalent to

$$\begin{aligned} 2|x| + |x| &= 3 \\ 3|x| &= 3 \\ |x| &= 1 \end{aligned}$$

is equivalent to

$(x = 1$ and $x \geq 0)$ or $(-x = 1$ and $x < 0)$

Thus, the truth set is $\{1, -1\}$

5. (a) $\frac{1}{x} + \frac{2}{3x} = \frac{1}{3}$ for $x = 5$

(b) $x + \sqrt{2} > \sqrt{2}$ for $x > 0$

(c) If $\sqrt{a^2 + b^2} = a + b$,

then

$$\begin{aligned} a^2 + b^2 &= (a + b)^2 \\ a^2 + b^2 &= a^2 + 2ab + b^2 \\ 0 &= 2ab \end{aligned}$$

But $a > 0$ and $b > 0$ so $2ab > 0$.

Thus $2ab = 0$ and $2ab > 0$ is a contradiction.

If $\sqrt{a^2 + b^2} > a + b$

then $a^2 + b^2 > (a + b)^2$

$a^2 + b^2 > a^2 + 2ab + b^2$

$0 > 2ab$

But $2ab < 0$ and $2ab > 0$ is a contradiction.

Thus $\sqrt{a^2 + b^2} < a + b$

Notice that the student was not asked to prove the relation, but we include the proof in case he insists on a proof.

(d) $\sqrt[4]{3^2} = \sqrt{3}$

(e) $(\sqrt{m} + \sqrt{n})(\sqrt{m} - \sqrt{n}) = (m - n)$ for $m > 0$ and $n > 0$

(f) $|x| + 5\sqrt[3]{125} > (-x^3)$. Notice that the left member is positive and the right member is negative.

(g) $\sqrt{\frac{3^2 \cdot 5^2}{256}} < 7\sqrt{\frac{1}{8}}$

6. $\sqrt{390} \approx 19.7$

7. $\sqrt{3900} \approx 62.45$

8. (a) 3^5
   (b) 6^2
   (c) $3^2 \cdot 2^3$
   (d) 3^4
   (e) 3^5
   (f) $3^2 + 2^2$

9. (a) $10^0 = 1$
   (b) 10^1
   (c) $10^n + 2$
   (d) 10^4
   (e) 10^{-5}
   (f) 10^{6n}

10. (a) $\frac{4}{5}$
   (b) $3b^2$
   (c) $\frac{m^3}{2q^3}$
   (d) $\frac{7}{5}$
   (e) $\frac{x + 1}{x - 1}$
   (f) $\frac{3}{2x}$

11. (a) all x such that $x < -6$
   (b) $\{\frac{4}{3}\}$

(c) all y such that $y < \frac{15}{8}$

(d) $\{\frac{3}{16}, -\frac{3}{16}\}$

(e) $\{2\}$

(f) $\{\frac{1}{2}\}$

12. $n^2 - n + 41$ fails to give a prime for $n = 41$, since then the sum of the last two terms is zero. This leaves n^2, which has n as a factor.

If an algebraic sentence is true for the first 400 values of the variable, it is not certain that it is true for the 401st.

13. The average of n numbers $a, b, c, \ldots$ is

$$\frac{a + b + c + \ldots \text{ to } n \text{ numbers}}{n}.$$

If g is the "guessed average", then the average of the differences is

$$\frac{(a - g) + (b - g) + (c - g) + \cdots \text{ to } n \text{ numbers}}{n}$$

$$= \frac{a+b+c+ \cdots \text{ to } n \text{ numbers} - ng}{n}$$ Commutative property of addition and the distributive property

$$= \frac{a+b+c+ \cdots \text{ to } n \text{ numbers}}{n} - g$$ Distributive property and the multiplication property of 1

When we add this average of the differences to our "guessed average" g, we have

$$\frac{a + b + c + \cdots \text{ to } n \text{ numbers}}{n} - g + g$$

$$= \frac{a + b + c + \cdots \text{ to } n \text{ numbers}}{n},$$

and this is the average.

$195 - 200 = -5$
$205 - 200 = 5$
$212 - 200 = 12$
$201 - 200 = 1$
$198 - 200 = -2$
$232 - 200 = 32$
$189 - 200 = -11$
$178 - 200 = -22$
$196 - 200 = -4$
$204 - 200 = 4$
$182 - 200 = -18$

Sum of the differences is -10.
Average of the differences $= -\frac{8}{11}$.

Adding this to 200 gives $199\frac{3}{11}$ for the team average.

14. If the rat weighs x grams at the beginning of the experiment it will weigh $\frac{5}{4}x$ grams after the rich diet and $\frac{3}{4}(\frac{5}{4}x)$ at the end of the experiment. Thus, the difference is $\frac{15}{16}x - x = -\frac{1}{16}x$ grams.

15. If x is the number of quarts of white paint, then $3x$ is the number of quarts of grey paint and

$$x + 3x = 7 \cdot 4$$
$$4x = 4 \cdot 7$$
$$x = 7 \qquad 3x = 21$$

Thus the man bought 1 gallon and 3 quarts of white and 5 gallons and 1 quart of grey. The information on the cost of the paint was unnecessary.

16. Proof: Either $\sqrt{a} < \sqrt{b}$, $\sqrt{a} = \sqrt{b}$, or $\sqrt{a} > \sqrt{b}$.

Assume $\sqrt{a} < \sqrt{b}$
then $a < b$ If $x < y$ then $x^2 < y^2$
$a < b$ and $a > b$ is a contradiction

Assume $\sqrt{a} = \sqrt{b}$
then $a = b$
$a = b$ and $a > b$ is a contradiction

Thus $\sqrt{a} > \sqrt{b}$

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Chapter 11

Suggested Test Items

1. Describe the set of numbers for which the following name real numbers.

   (a) $\sqrt{a}$

   (b) $\sqrt{x^2}$

   (c) $\sqrt[3]{b}$

   (d) $\sqrt{\frac{1}{a}}$

   (e) $\sqrt[4]{x^4}$

   (f) $\sqrt{1 + m}$

2. Simplify

   (a) $\sqrt{27}$

   (b) $\sqrt[3]{8}$

   (c) $\sqrt{\frac{4}{3}}$

   (d) $\sqrt{24} - \frac{1}{3}\sqrt{54}$

   (e) $\frac{\sqrt{18}}{\sqrt{3}}$

   (f) $\sqrt{2}\ \sqrt{18}$

3. Simplify assuming that all variables represent positive numbers.

   (a) $\sqrt{\frac{1}{2x}}$

   (b) $\sqrt{\frac{1}{ab}}\cdot\sqrt{a^3b}$

   (c) $\dfrac{\sqrt{\frac{x}{y^2}}}{\sqrt{\frac{y}{x^2}}}$

   (d) $\sqrt{\frac{2a}{3}}\cdot\sqrt{\frac{3a}{4}}\sqrt{\frac{4a}{5}}\sqrt{\frac{5a}{6}}$

   (e) $\sqrt{4a^3} + \sqrt{\frac{4}{9a}}$

   (f) $\sqrt{2}(\sqrt{2} - \sqrt{18})x$

4. (a) If $\sqrt{a}$ is a rational number, what kind of number is a?

   (b) If a and b are different positive primes, what kind of number is $\sqrt{ab}$?

5. If $\sqrt{85} \approx 9.219$ and $\sqrt{8.5} \approx 2.939$, find approximations for

   (a) $\sqrt{0.0085}$

   (b) $\sqrt{850}$

   (c) $\sqrt{85000}$

   (d) $\sqrt{.85}$

6. Explain, using an example, that $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ is not a sentence for $a < 0$ and $b \leq 0$.

7. Explain why $\sqrt{x^3}$ is meaningful only if $x \geq 0$.

8. Is $\sqrt{1764}$ rational or irrational? Explain.

9. Simplify each of the following:

   (a) $\sqrt{15}\ \sqrt{20}$

   (b) $\frac{\sqrt{15}}{\sqrt{20}}$

   (c) $\frac{\sqrt{20}}{\sqrt{15}}$

   (d) $\sqrt{20} + \sqrt{15}$

Chapter 12

POLYNOMIAL AND RATIONAL EXPRESSIONS

In this chapter, factoring of expressions is introduced by using the analogy with factoring of positive integers as motivation. The way was prepared for this in Chapter 10. An important point is that factoring of rational numbers becomes significant only when the problem is restricted to the integers. In factoring expressions, the class of rational expressions corresponds to the system of rational numbers, and the class of polynomials corresponds to the system of integers. The intended implication here is that the rational expressions do constitute an algebraic system with the polynomials as a sub-system.

Although the following discussion is not for student consumption, it is important for the teacher to understand the algebra of expressions.

Consider, for example, the distributive property:

$$a(b + c) = ab + ac$$

We have always understood a, b, c to be real numbers, so that we are dealing with an assertion about real numbers. The assertion involves two phrases "$a(b + c)$" and "$ab + ac$" and enables us to replace either phrase by the other, in any statement about real numbers, without altering the validity of the statement. However, suppose we forget, for the moment, that we are talking about real numbers (as was commonly done at one time in elementary algebra). Then the distributive property (or "law") becomes a "rule" for transforming algebraic expressions, that is, a rule in the "game" of "symbol pushing". From this point of view, the various fundamental properties, with which we have been working, constitute the complete set of rules of the game. Attention is thus shifted from the system of real numbers to the language used to talk about the real numbers. Although blind symbol pushing is highly undesirable, it is a fact that we do work with expressions from this point of view. This is what we are doing whenever we discuss the form of an expression. The difference is that symbol pushing at this level is not mechanical but is with reference to an algebraic system. We shall now describe more carefully this system.

In the first place, we "add" and "multiply" expressions by use of what we have called "indicated" sums and products. Thus, if A and B are expressions then $A + B$ and $A \cdot B$ are also expressions. We also write $A = B$ provided for each permitted value of each variable involved in A and B, the numerals "A" and "B" name the same number. This is actually a definition of equality for expressions. In some books, this kind of equality is called identity. With these agreements, the following basic properties could be found for expressions and have, in fact, been used many times in the course:

1. If A, B are expressions, then $A + B$ is an expression
2. If A, B are expressions, then $A + B = B + A$.
3. If A, B, C are expressions, then $(A + B) + C = A + (B+C)$.
4. There is an expression 0 such that $A + 0 = A$ for every A.
5. For each expression A, there is an expression $-A$ such that $A + (-A) = 0$.
6. If A,B are expressions, $A \cdot B$ is an expression.
7. If A, B are expressions, $AB = BA$.
8. If A, B, C are expressions, then $(AB)C = A(BC)$.
9. There is an expression 1 such that $A \cdot 1 = A$ for every A.
10. For each expression A different from 0, there is an expression $\frac{1}{A}$ such that $A \cdot \frac{1}{A} = 1$.
11. If A, B, C are expression, then $A(B + C) = AB + AC$.

Thus we see that the class of expressions satisfies the axioms for a field. The smaller class, consisting of just the rational expressions, also satisfies these properties. The class of all polynomials (or all polynomials in one variable over the integers) is a sub-system of the class of rational expressions and has all of these properties except Number 10. Notice also that the rational numbers satisfy all of these properties, and the integers satisfy all except Number 10-- hence, the parallel between rational expressions and polynomial, on the one hand, with rational numbers and integers on the other.

Once these general properties are established, we can study rational expressions and polynomials as algebraic systems in their own right independently of their connection with real numbers. This is symbol pushing par excellence. Our work with factoring, simplification of rational expressions and division of polynomials forms a small fragment of the study of these general systems, although we have not presented it explicitly as such. This way of looking at the language of algebra, which is implicit in much of what has gone before and actually comes out into the open in the present chapter, will turn up frequently in later courses in algebra. A good student automatically shifts to this point of view about algebra as he matures. However, if this occurs before he understands, at least intuitively, that an algebraic system is involved, only confusion will result. This is why it is important to go back to the real numbers whenever students show signs of mechanical manipulation of symbols. For further discussion see Studies in Mathematics, Volume III, pages 6.1-6.8.

12-1 Polynomials and Factoring.

Most of the work in the chapter is with polynomials over the integers. The definition of polynomial over the integers and a statement of the problem of factoring are given in this section. The problems in this section are primarily concerned with bringing out these ideas rather than with developing the techniques of factoring. The latter are dealt with in the next five sections.

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Page 314. If we were being very careful in our language, we should replace the word "integers" by "numerals for integers" in the definition of polynomial over the integers. Since there can be no doubt about what is intended here, we choose to keep the definition as simple as possible.

The usual definition of polynomial (in one variable x) is that it is an expression of the form

$$a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0.$$

With this definition, the expression $(x^2 - 1)(3x - 5)$ is not a polynomial but is an indicated product of polynomials. On the other hand, we usually want to call this a polynomial because we have in mind the fact that it can be written as a polynomial:

$$(x^2 - 1)(3x - 5) = 3x^3 - 5x^2 - 3x + 5.$$

This sentence can be thought of as a defintion of multiplication for polynomials, as defined above, since it specifies what polynomial is indicated by the given product. Addition of polynomials can be looked at in a similar way.

In our development it was more natural to regard a much wider class of expressions as polynomials. Then the definitons of addition and multiplication are obvious, and an equation much as the above may be thought of as a definition of equality.

These two points of view, although conceptually different, amount in practice to exactly the same thing. The only real difference is in the way we think about rather than the way we work with expressions. Thus we can always simplify one of our polynomials (in one variable) to the special form indicated above and think of the given polynomial as represented by its simplified form. In addition to lending itself better to the informal treatment of polynomials which we wanted to give, the definition we use also makes it easier to discuss polynomials in several variables.

Answers to Problem Set 12-1a; pages 315-316 :

1. (a), (c), (d), (e) are polynomials over the integers
   (b), (f) are polynomials over the rational numbers. (g) is a polynomial over the reals. } See section 12-6.
   (h) is not a polynomial at all. However, notice that $|x| + 1$ can be described using polynomials:

$$|x| + 1 = \begin{cases} x + 1, & \text{if } x \geq 0 \\ -x + 1, & \text{if } x < 0 \end{cases}$$

A better example here is the expression, $(|x|)^2$, which is not a polynomial but can be written as a polynomial: $(|x|)^2 = x^2$. This is analogous to $\frac{x(x^2 + 1)}{x^2 + 1}$ which is not a polynomial but can be written as a polynomial, namely, x.

2. (a), (c), (d) are polynomials over the integers.
   (b) is a polynomial over the rationals.
   (e), (f), (g) and (h) are not polynomials of any kind under our definition. However, since $\frac{x + y}{2}$ means the same as $(x + y)\frac{1}{2}$, it would not be objectionable to admit (f) as a polynomial over the rationals. The point here is that $\frac{x + y}{2} = (x + y)\cdot\frac{1}{2}$, by definition, rather than by use of any properties of real numbers. On the other hand, we would not wish to regard (h) as a polynomial in spite of the fact that it can be written as a polynomial:

$$\frac{3s(u + v)}{s} = 3(u + v).$$

3. (a) $2x^2 - 4x$ (e) $u^2 - \frac{1}{4}$

   (b) $x^2y - 2xy^2$ (f) $x^2 + 4x + 4$

   (c) $t^2 + t - 6$ (g) $18t^2 - 15t - 88$

   (d) $-\frac{9}{4}x^3y^3z$ (h) $y^2 + y - 2$

   All are polynomials over the integers except (d) and (e) which are over the rational numbers.

4. (a) 0 (d) $3u^2 + u - uv$

   (b) $a^2 - 2$ (e) $2 + 2s - 6st$

   (c) $3su + 3sv$ (f) $2x^2 - 3xy - 2y^2 + 5y - 2$

   All are polynomials over the integers although the two factors in (b) are not over the integers.

5. Yes! We hope that this will be obvious to everyone. The point in mentioning it is to suggest that we are dealing with a system.

6. No! However, it may be possible to write such a quotient as a polynomial. See, for example, part (h) of Problem 2. The idea here is to suggest the fact that polynomials as a system are not closed under division although some rational expressions can be written as polynomials.

Answers to Problem Set 12-1b; pages 318-319:

1. (a), (b), (e), (f).

   (c) is a case of factoring $s^2 - 3$ as a polynomial over the real numbers.

   (d) is a case of factoring $3t - 5$ as a polynomial over the rational numbers.

2. (c), (d), (f)

   In (a), (b) and (e), polynomials over the integers are factored but the factors are not even polynomials.

[pages 315-316, 318]

3. (a)

   (b) a polynomial over the rational numbers is factored into polynomials over the rational numbers.

   (c) and (d) are not factored at all.

   (e) The second factor is not a polynomial.

4. 2(e) $(|t| + 1)(|t| - 1) = (|t|^2 - 1) = t^2 - 1$

5. (a), (e)

6. (a) $a(a + 2b)$

   (b) $3(t - 2)$

   (c) $a(b + c)$

   (d) $3xz\,(x - y)$

   (e) $a(x - y)$

   (f) $6(p - 2q + 5)$.

7. (a) $z^2(z + 1)$

   (b) $15\,(a^2 - 2b)$

   (c) $x^2(1 - x^2)$ is the result expected. Some may obtain $x^2(1 + x)(1 - x)$, which is technically correct since the distributive property is certainly involved.

   (d) $a(a^2 - 2a + 3)$

   (e) $6(x^2 - 24y - 25)$ is expected.

   (f) $y\big(3x + (x - 3)\big)$ or $y(4x - 3)$.

8. (a) $2\big((z + 1) - 3zw\big)$

   (b) $a^2b^3(a + b - 1)$

   (c) No factoring possible

   (d) $ab(x - y)$

   (e) $6r^2s(x - y)$

   (f) $(u^2 + v^2)(x - y)$

   (g) $(x - y)(4x - y)$

   (h) $2^2 3^2 a^2 b^2 c^2$

9. (a) 1, 1, 2

   (b) 2, 1, 3

   (c) 3, 2, 5

   (d) 0, 5, 5

   (e) 2, 4

   (f) The degree of the product is equal to the sum of the degrees of the factors.

12-2. Factoring by the Distributive Property.

All non-trivial factoring of polynomials involves the distributive property as well as the various other properties of real numbers. However, the problems in this section involve the distributive property in an especially straightforward and explicit way. This factoring is often referred to as "removing common (monomial, etc.) factors."

Example 4. There is an opportunity in this example to emphasize one of the most important uses of factoring, namely, to solve polynomial equations. Since students like to solve equations, this should help stimulate their interest in factoring.

Answers to Problem Set 12-2a; pages 321-322:

1. $3xz(2x - y)$ over the integers
2. $3st(3 - u)$ over the integers
3. $36(4x^2 - 6s + 5y)$ over the integers
4. $\frac{3v}{5}(2u^2 - 3uv + 5v)$
5. $-xy^2(x^2 - 2x - 1)$ over the integers
6. $\frac{1}{36}ab(6 + 10a - 21b)$
7. $s\sqrt{3}(1 + s\sqrt{2})$
8. $\frac{a}{6\sqrt{2}}(3a - 4b)$
9. no common factor
10. $(a + 3)(x - 1)$ over the integers
11. $(x + 3)(x + 1)$ over the integers
12. $(u + v)(x - y)$ over the integers
13. $(a - b)(a + b)$ over the integers
14. $(x + y)u$ over the integers

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15. 0 over the integers
16. $(x + y)(3x - 5y + 1)$ over the integers
17. $3a\sqrt[3]{2}(2 + 5b)$
18. $|x|(3 + 2a)$
19. $7y|x|(1 - 3y)$
20. $(u + v)(r - s)$ over the integers
21. $(a + b + c)(x - y)$ over the integers
22. $(a + b + c)x$ over the integers

Page *323*. It is necessary to emphasize that factoring involves writing the given polynomial as a product of polynomials. Sums of products do not count.

Answers to Problem Set 12-2b; pages 324-325:

1. $a(x + 2) + 3(x + 2) = (a + 3)(x + 2)$
2. $x(u + v) + y(u + v) = (x + y)(u + v)$
3. $a(2b + a) + 1(2b + a) = (a + 1)(2b + a)$
4. $3s(r - 1) + 5(r - 1) = (3s + 5)(r - 1)$
5. $x(5 + 3y) - 1(5 + 3y) = (x - 1)(5 + 3y)$
6. 0
7. $a(a - b) + c(a - b) = (a + c)(a - b)$
8. $t(t - 4) + 3(t - 4) = (t + 3)(t - 4)$
9. not factorable
10. $(2a - 3b)(a - b\sqrt{3})$
11. $3x(5a + 4b - 3c + 2d)$
12. $2(a - b) + u(a - b) + v(a - b) = (2 + u + v)(a - b)$
13. $x(u + v - w) + y(u + v - w) = (x + y)(u + v - w)$
14. $a(a - 4x) + 2b(a - 4x) + 3c(a - 4x) = (a + 2b + 3c)(a-4x)$

15. $\frac{a}{6}(3xy - 6ay + bx - 2ab) = \frac{a}{6}\left(3y(x - 2a) + b(x - 2a)\right) =$

$\frac{a}{6}(3y + b)(x - 2a)$

16. $x^2 + 3x + x + 3 = x(x + 3) + 1(x + 3) = (x + 1)(x + 3)$

17. $a^2 - ab + ab - b^2 = a(a - b) + b(a - b) = (a + b)(a - b)$

12-3. Difference of Squares.

Answers to Problem Set 12-3; pages 327-329:

1. (a) $a^2 - 4$

(b) $4x^2 - y^2$

(c) $m^2n^2 - 1$

(d) $9x^2y^2 - 4z^2$

(e) $a^4 - b^4$

(f) $x^2 - a^2$

(g) $2x^2 + 3xy - 2y^2$

(h) $r^3 + r^2s^2 - rs - s^3$

2. (a) $(2x - 1)(2x + 1)$

(b) $9(3 - y)(3 + y)$

(c) $(a - 2)(a + 2)$

(d) $(1 - n)(1 + n)$

(e) $(5x - 3)(5x + 3)$

(f) $4(2x - y)(2x + y)$

3. (a) $(5a - bc)(5a + bc)$

(b) $5(2s - 1)(2s + 1)$

(c) $6(2y - z)(2y + z)$

(d) $4x(2x - 1)(2x + 1)$

(e) $4(2x - 1)(2x + 1)$

(f) $(7x^2 - 1)(7x^2 + 1)$

4. (a) $(x - 2)(x + 2)$

(b) not factorable over the integers

(c) $(x^2 + 2)(x^2 - 2)$

(d) not factorable over the integers

(e) $3(x - 1)(x + 1)$

(f) $(4x^2 + 1)(2x - 1)(2x + 1)$

5. (a) $(a - 2)a$

(b) $2a - 3$ not factorable over the integers

(c) $4\ mn$

(d) 0

(e) $(x - y)(x + y - 1)$

(f) $(x - y)(1 - x - y)$

6. (a) $x^2 - 9 = 0$

$(x - 3)(x + 3) = 0$

is equivalent to

$x - 3 = 0$ or $x + 3 = 0$

The truth set is $\{-3, 3\}$

(b) $\{\frac{1}{3}, -\frac{1}{3}\}$

(c) $\{\frac{1}{5}, -\frac{1}{5}\}$

(d) $\{2, -2\}$

(e) $\{0, \frac{1}{2}, -\frac{1}{2}\}$

(f) $\emptyset$

(g) $\{2, -2\}$

(h) $\{1, -5\}$

7. (a) 396

(b) 1591

(c) $884r$

(d) $391xy$

(e) 9999

(f) $2000\ mn$

(g) $1584m^2 - 1584n^2$

(h) 1584

8. (a) $899 = 30^2 - 1 = (30 - 1)(30 + 1)$, thus 899 is factorable.

(b) $1591 = 40^2 - 3^2 = (40 - 3)(40 + 3)$, thus 1591 is factorable.

(c) $391 = 20^2 - 3^2 = (20 - 3)(20 + 3)$, thus 391 is factorable.

(d) $401 = 20^2 + 1$, can not tell by difference of squares factoring. By prime factorization methods we see that 401 is not divisible by 2,3,5,7,11,13,17, or 19. Thus 401 is prime.

*9. The reciprocal of $2 - \sqrt{3}$ is $2 + \sqrt{3}$ and vice versa. Here is an example of a pair of numbers which are reciprocals but one is not the other "turned upside down".

(a) $\frac{2}{23}(5 - \sqrt{2})$

(b) $\frac{7 + 3\sqrt{5}}{2}$

(c) $-\frac{3}{2}$, notice $\sqrt{4}$ is rational

(d) $3\sqrt{2} + \sqrt{15}$

*10. (b) $(t + 1)(t^2 - t + 1)$

(c) $(s + 2)(s - 2s + 4)$

(d) $(3x + 1)(9x^2 - 3x + 1)$

*11. (b) $(t - 1)(t^2 + t + 1)$

(c) $(s - 2)(s^2 + 2s + 4)$

(d) $(2x - 1)(4x^2 + 2x + 1)$

12-4. Perfect Squares.

Page 330. Example 1. A quadratic polynomial $x^2 + px + q$ where p and q are integers is a perfect square if and only if

(1) q is the square of an integer m.

(2) either $p = 2m$ or $p = -2m$.

Answers to Problem Set 12-4a; pages 331-332:

1. (a) 9

(b) 16

(c) 36

(d) t^2

(i) 5

(j) $2\sqrt{3}$, zero should also be accepted.

(k) $2\sqrt{35}$, 3 could be accepted

(ℓ) $4t^2$

(e) x^2

(f) $10u$

(g) $12s$

(h) 9

(m) $16v^2$

(n) $56xy$

(o) 4

(p) $6(x - 1)$

2. (a), (c), (d), (e), (g), and (h) are perfect squares

3. (a) $(a - 2)^2$

(b) $(2x - 1)^2$

(c) $(x - 2)(x + 2)$

(d) not factorable over the integers.

(e) $(2t + 3)^2$

(f) $7(x + 1)^2$

(g) not factorable over the integers

(h) $(2z - 5)^2$

(i) not factorable over the integers

(j) $(3a - 4)(3a - 2)$

(k) $5(4 - x)$

(ℓ) not factorable over the integers.

(m) not factorable over the integers.

(n) not factorable over the integers.

(o) $2a(a - 5b)^2$

(p) $(s + 5)^2$

(q) $(t - s - 2)(t + s)$

(r) $(x - 1)^2(x + 1)^2$

(s) $(z^2 + 8)^2$

4. (a) $x^2 + 6x + 9$

(b) $x^2 - 4x + 4$

(c) $x^2 + 2\sqrt{2}\,x + 2$

(d) $a^2 + 2ab + b^2$

(e) $x^2 - 2xy + y^2$

(f) $x^2 - 2x + 1 - a^2$

(g) $x^2 - 2x + 1 - a^2$

(h) $5 + 2\sqrt{6}$

(i) $100^2 + 2 \cdot 100 \cdot 1 + 1 = 10,201$

Page 333. Example 4. The method of completing the square is touched on again in Section 12-6 for polynomials over the real numbers. It is also used in Chapters 16 and 17 in connection with graphing quadratic polynomials and functions.

Example 5. Notice that we have proved that the truth set of this equation is empty. In other words, we do not conclude that it is empty just because we are unable to find solutions by our methods of factoring.

This example points out the connection between "factoring a polynomial over a set" and "solving the corresponding polynomial equation". If a polynomial equation has solutions which are integers, then the polynomial over the integers can be factored, and conversely. If the polynomial equation has solutions which are real numbers, then the polynomial over the real numbers can be factored, and conversely. Thus, if a polynomial equation has an empty truth set, the polynomial cannot be factored. The conclusion in Example 5 is that $x^2 - 8x + 18$ cannot be factored over the real numbers.

Answers to Problem Set 12-4b; page 334:

1. (a) $x^2 + 4x + 4 - 1 = (x + 2)^2 - 1$

$$= (x + 2 - 1)(x + 2 + 1)$$

$$= (x + 1)(x + 3)$$

(b) $(x - 4)(x - 2)$

(c) $(x - 4)(x + 2)$

(d) $(x - 6)(x - 4)$

(e) $(x - 12)(x + 2)$

(f) $(x - 1)^2 - 4(x - 1) + 4 - 9 = \left((x - 1) - 2\right)^2 - 3^2$

$$= (x - 3)^2 - 3^2$$

$$= (x - 3 - 3)(x - 3 + 3)$$

$$= (x - 6)x$$

2. (a) $p = 1$ (d) none

(b) $p = 8$ or -8 (e) $p = 4$

(c) all

3. (a) $y^2 - 10y + 25 = 0$

$$(y - 5)^2 = 0$$

is equivalent to

$y - 5 = 0$ or $y - 5 = 0$

The truth set is $\{5\}$

(b) $\{\frac{5}{2}\}$

(c) $9a^2 + 6a + 4 = 0$

$9a^2 + 6a + 1 + 3 = 0$

$(3a + 1)^2 + 3 = 0$

$(3a + 1)^2$ is ≥ 0 for every a.

Thus the truth set is empty.

(d) The truth set is $\{2\}$.

(e) The truth set is $\{0,2\}$.

(f) The truth set is $\emptyset$.

(g) The truth set is $\{4, -2\}$.

(h) The truth set is $\{4, 6\}$.

12-5 Quadratic Polynomials.

This section, along with the preceding three sections, covers the standard techniques of factoring found in the usual elementary algebra course. Although these techniques are important, they are not ends-in-themselves. The ideas behind factoring should be brought to the student's attention at every opportunity. The text contains many exercises designed to sharpen the student's factoring techniques. The ones you assign should be selected with care. Do not swamp the students in a deluge of drill problems and do not avoid the "idea" problems. It is a temptation to treat factoring in a mechanical way and so allow the students to fall into the trap of blind symbol pushing. This danger is always present when techniques are emphasized, and it is up to the teacher to maintain the proper perspective in these situations.

Page 337. Example 3. $x^2 - 10x + 36$ is also not factorable because 10 is too small. The smallest positive value of p for which $x^2 + px + 49$ is factorable is 14 and this gives a perfect square. $x^2 + 14x + 49 = (x + 7)^2$. $x^2 + 13x + 49$ is not factorable because 13 is too small.

The polynomial $x^2 + 40x + 36$ is not factorable because 40 is too large. Similarly, $x^2 - 38x + 36$ is not factorable because 38 is too large. $x^2 - 50x + 49 = (x - 1)(x - 49)$. 50 is the largest absolute value p can have in order for $x^2 + px + 49$ to be factorable. See Problems 4, 5, 6, below.

Answers to Problem Set 12-5a; pages 337-339:

1. (a) $(a + 5)(a + 3)$
   (b) $(a - 5)(a - 3)$
   (c) $(a + 5)(a - 3)$
   (d) $(a - 5)(a + 3)$
2. (a) $(t + 10)(t + 2)$
   (b) $(t + 20)(t + 1)$
   (c) $(t + 5)(t + 4)$
   (d) not factorable over the integers

3. (a) $(a + 11)(a - 5)$
   (b) $(x - 3(x - 2)$
   (c) $(u - 6)(u - 4)$
   (d) $(y - 18)(y + 1)$
   (e) not factorable over the integers

4. (a) $-(x - 4)(x - 3)$
   (b) $-(x + 12)(x - 1)$
   (c) $-(x + 6)(x - 2)$
   (d) $-(x + 12)(x + 1)$
   (e) not factorable over the integers

5. (a) $(a - 8)^2$
   (b) not factorable over the integers, $8 < 2\sqrt{64}$
   (c) not factorable over the integers. Since 2 is a factor of 36 and $64 = 2^6$, the 2's must be split. $32 + 2 < 36$ and any other split gives smaller sums.
   (d) $(a - 16)(a - 4)$
   (e) not factorable over the integers.

6. (a) $(x - 3)(x + 3)$
   (b) not factorable over the integers
   (c) not factorable over the integers
   (d) $(h - 13)(h + 13)$

7. (a) $(z^3 - 8)(z^3 + 1)$ or

   $$^{*}\,(z - 2)(z^2 + 2z + 4)(z + 1)(z^2 - z + 1)$$

   (b) $(b^2 - 7)(b - 2)(b + 2)$
   (c) $(a - 3)(a + 3)(a - 2)(a + 2)$
   (d) $(y - 3)(y + 3)(y^2 + 9)$

8. (a) $(a + 7)(a - 2)$
   (b) not factorable over the integers.
   (c) $(a - 12)(a - 9)$

   $108 = 2^2 \cdot 3^3$

   3 is a factor of 21 but 2 is not, so the 3's are split but the 2's remain in one factor. Thus

   $(3 \cdot 2^2) + (3^2) = 21$
   (d) $(a + 40)(a - 15)$

   $600 = 2 \cdot 3 \cdot 5^2$

   Of 2,3, and 5 only 5 is a factor of 25 so the 5's are split but the 2's must be in the same factor.

   Thus $(2^3 \cdot 5) - (5 \cdot 3) = 25$

9. (a) $3(y^2 - 4y + 4) = 3(y - 2)^2$
   (b) $x(x^2 + 19x + 34) = x(x + 17)(x + 2)$
   (c) $5a(a^2 - 3a + 6)$
   (d) $7(x - 3)(x + 3)$

10. (a) The truth set is $\{12, -3\}$
    (b) The truth set is $\{3,2\}$
    (c) The truth set is $\{4,9\}$
    (d) The truth set is $\{0, -6\}$
    (e) The truth set is $\{6,1\}$
    (f) The truth set is $\{3, -2\}$
    (g) The truth set is $\{-4,3\}$
    (h) The truth set is $\emptyset$.

11. (a) If x is the number, then $x^2 = 6x + 7$
The truth set is $\{7, -1\}$
The number could be either 7 or -1.

(b) If the width of the rectangle is w inches then the length is $w + 5$ inches and $w(w + 5) = 84$.
The truth set is $\{-12, 7\}$.
Thus the width of the rectangle is 7 inches.

(c) The number is either 1 or 9.

12. If the length of the bin is x feet, then the width is $12-x$ feet and $70 = x(12-x)2$.
The length of the bin is 7 feet and the width is 5 feet.

13. The square is 6 feet on a side and the rectangle is 12 feet long and 3 feet wide.

*14. Assume that $(x + m)(x + n) = x^2 + px + q$.
Then, $mn = q$ and $m + n = p$.
Also, $(x - m)(x - n) = x^2 - (m + n)x + mn = x^2 - px + q$.

*15. $x^2 + px + 36$ is a perfect square for $p = 12$ or $p = -12$. 12 is the smallest value $|p|$ can have for $x^2 + px + 36$ to be factorable. Values of p for which $x^2 + px + 64$ is factorable are obtained as follows:
Note that $p = m + n$ where $mn = 2^6$.

$m \cdot n$	$m + n$
$2^6 \cdot 1$	65
$2^5 \cdot 2$	34
$2^4 \cdot 2^2$	20
$2^3 \cdot 2^3$	16

Positive values of p are $16, 20, 34, 65$ and negative values are $-16, -20, -34, -65$. The perfect squares are given by $p = 16$ or $p = -16$. Note that 16 is the smallest value $|p|$ can have.

The student should be able to generalize the results in the two examples and guess that $2n$ is the smallest positive value of p for which $x^2 + px + n^2$ is factorable and that this gives the perfect square, $(x + n)^2$.

The largest value of p for which $x^2 + px + n^2$ is factorable is $n^2 + 1$, in which case

$$x^2 + px + n^2 = (x + 1)(x + n^2)$$

The above results are special cases of the following general theorem, whose proof is too difficult for all but the best students.

Theorem. Consider the quadratic polynomial $x^2 + px + q$, where p and q are positive integers. Then,

(1) The largest value of p for which the polynomial is factorable is $q + 1$.

(2) The smallest value of p for which the polynomial is factorable is equal to $m + n$, where $mn = q$ and m,n are as nearly equal as possible.

Proof: Assume that $(x + m)(x + n) = x^2 + px + q$. Then,

$m + n = p$ and $mn = q$. Observe that

$(m + n)^2 - (m - n)^2 = 4\ mn$ for all values of m and n.

Therefore, $p^2 = 4q + (m - n)^2$.

It follows that p^2, and hence, p, will have its greatest value when $(m - n)^2$ is as large as possible. This obviously occurs with the factorization $q = q \cdot 1$, giving $p = m + n = q + 1$.

Similarly, p^2, and hence, p, will have its smallest value when $(m - n)^2$ is as small as possible, that is, when m and n are as nearly equal as possible.

Answers to Problem Set 12-5b; pages 343-346:

1. (a) $(x + 1)(2x + 3)$
   (b) $(2x + 1)(x + 3)$
   (c) not factorable over the integers. The only factors we have to work with are 1, 2, and 3 so the maximum sum is 7.

2. (a) $(3a + 7)(a - 1)$
   (b) $(3a - 7)(a + 1)$
   (c) $-(3a + 7)(a - 1)$

3. (a) $(4y - 1)(y + 6)$
   (b) $(x + 8)(x - 4)$
   (c) $(4a - 1)(2a + 3)$

4. (a) not factorable over the integers
   (b) $(3x + 1)(x - 6)$
   (c) $3(y^2 + y - 2) = 3(y + 2)(y - 1)$

5. (a) $(3x - 2)(3x + 2)$
   (b) $(3x - 2)^2$
   (c) $(3x + 2)^2$

6. (a) $(3a + 2)(3a - 1)$
   (b) $3a(3a + 1)$
   (c) $9(a^2 + 1)$

7. (a) $3(x - 3)(4x - 5)$
   (b) $(5x + 9)(2x + 5)$
   (c) $(2x - 15)(5x + 3)$

8. (a) $(6 + a)(1 - 4a)$
   (b) $-(3x + 1)(x - 6)$
   (c) $(7x - 2)(x + 3)$

9. (a) $(p + q)^2$
   (b) $(2a - b)(2a - 7b)$
   (c) $(5x - 7y)^2$

10. (a) $2a^2(a^2 + 10a + 25) = 2a^2(a + 5)^2$

(b) $b(a^2 - 9a + 25)$, $a^2 - 9a + 25$ is not factorable since $9 < 2\sqrt{25}$

(c) $(2a + 5)(a + 5)$

11. (a) $6(x - 25)(x + 1)$

(b) $(x - 6)(6x + 25)$

(c) $6(x + 5)^2$

(d) $(x - 6)(6x - 25)$

(e) $6x^2 + 25x + 150$ is not factorable over the integers

(f) $(3x + 10)(2x + 15)$

(g) $3(x - 2)(2x - 25)$

(h) $3(x - 2)(2x + 25)$

12. No. There is only one factor 2 in the coefficient of x^2 and none in the constant term. Therefore, either the inside product or the outside product will have a factor of 2 but not both. Thus, the sum of the inside and outside products will be odd.

13. Yes. $3x^2 + 5x - 12 = (3x - 4)(x + 3)$

14. (a) $8x^2 + 10x - 3 = 0$

$(4x - 1)(2x + 3) = 0$

is equivalent to

$4x - 1 = 0$ or $2x + 3 = 0$

$x = \frac{1}{4}$ or $x = -\frac{3}{2}$

The truth set is $\{\frac{1}{4}, -\frac{3}{2}\}$

(b) The truth set is $\{\frac{1}{3}, -\frac{1}{2}\}$

(c) The truth set is $\{-\frac{1}{3}, \frac{7}{2}\}$

(d) $a^2 - 4a + 15$ is not factorable over the integers. If you write

$$a^2 - 4a + 4 - 4 + 15 = 0$$

$$(a - 2)^2 + 11 = 0$$

it become clear that the truth set is empty.

15. (a) The truth set is $\{0, \frac{4}{9}\}$

(b) The truth set is $\{\frac{2}{3}, -\frac{2}{3}\}$

(c) The truth set is $\{-1, 3\}$

(d) The truth set is $\{\frac{1}{3}, \frac{5}{3}\}$

16. (a) $(w - 4)(w + 4)$

(b) $(x - 1)(x + 7)$

(c) $(y - 1)(y + 7)$

(d) $(a - 5 - 3b)(a - 5 + 3b)$

17. If x is one of the numbers, then $15 - x$ is the other number and

$$x^2 + (15 - x)^2 = 137$$

$$x^2 + 225 - 30x + x^2 = 137$$

$$2x^2 - 30x + 88 = 0$$

$$2(x^2 - 15x + 44) = 0$$

$$2(x - 4)(x - 11) = 0$$

The truth set is $\{4, 11\}$

The two numbers are 4 and 11.

18. If the width of the rectangle is w inches, its length is $w + 7$ inches and $w^2 + (w + 7)^2 = 13^2$

$$w^2 + (w + 7)^2 = 13^2$$

The truth set is $\{5, -12\}$.

The width of the rectangle is 5 inches.

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19. If n is one of the numbers, $n - 8$ is the other number and

$$n(n - 8) = 84 .$$

The truth set is $\{-6, 14\}$.

The two numbers are 6 and 14 or -6 and -14.

20. If q is one odd number, then $q + 2$ is the consecutive odd number, and

$$q(q + 2) = 15 + 4q .$$

The truth set is $\{-3, 5\}$.

The numbers are 5 and 7 or -3 and -1.

21. If Jim walked at the rate of x miles per hour Bill walked at the rate of $x + 1$ miles per hour. In one hour Jim walked $x \cdot 1$ miles and Bill walked $(x + 1) \cdot 1$ miles. Then

$$(x \cdot 1)^2 + \left((x + 1) \cdot 1\right)^2 = 5^2 .$$

The truth set is $\{-4, 3\}$.

Jim walked at the rate of 3 miles per hour and Bill walked at the rate of 4 miles per hour.

22. If the length of the base of the triangle is b inches, its altitude is $b - 3$ inches and

$$\frac{1}{2}(b)(b - 3) = 14 .$$

The truth set is $\{7, -4\}$.

The length of the base of the triangle is 7 inches.

23. If the width of the rectangle is w feet and

$$w(14 - w) = 24$$

The truth set is $\{12, 2\}$.

The width of the rectangle is 2 feet and the length is 12 feet.

12-6. Polynomials over the Rational Numbers or the Real Numbers.

The main idea here is that a polynomial, which is not factorable when considered as a polynomial over the integers, maybe factorable when considered as a member of the wider class of all polynomials over the real numbers. In other words, factoring depends on the class of polynomials we have under discussion.

This is a good place to raise the question of whether or not the polynomial $x^2 + 1$ can be factored if we allow a wider class of polynomials. It can be pointed out that, in order to factor $x^2 - 2$, we had to pass from the rational numbers to the real numbers and, in order to factor $x^2 + 1$, we must pass from the real numbers to the complex numbers, which will be studied in a later course. Note also the corresponding problem of solving equations. The equation $x^2 - 2 = 0$ does not have solutions if only rational numbers are permitted, but does have solutions if real numbers are allowed. Similarly, the equation $x^2 + 1 = 0$ does not have real number solutions but does have complex number solutions.

Answers to Problem Set 12-6; pages 348-351:

1. (a) $\frac{2}{3}a^2 - \frac{4}{3} = \frac{2}{3}(a^2 - 2)$ over the rational numbers

$= \frac{2}{3}(a - \sqrt{2})(a + \sqrt{2})$ over the real numbers

(b) $17u(1 - 3u^2)$ over the rational numbers

$17u(1 - \sqrt{3}u)(1 + \sqrt{3}\,u)$ over the real numbers

(c) $\frac{1}{2}t^3 - 3t^2 + 4t = \frac{1}{2}t(t^2 - 6t + 8)$

$= \frac{1}{2}t(t - 4)(t - 2)$ over the rational numbers.

(d) $\frac{1}{2}t^3 - 4t^2 + 8t = \frac{1}{2}t(t^2 - 8t + 16)$

$= \frac{1}{2}t(t - 4)^2$ over the rational numbers

(e) $a^4 - 16 = (a^2 - 4)(a^2 + 4)$

$= (a - 2)(a + 2)(a^2 + 4)$ over the rational numbers

(f) $4x^2 + 9$ is not factorable over the reals.

2. (a) $2x^2 - 6 = 0$

$2(x^2 - 3) = 0$

$2(x - \sqrt{3})(x + \sqrt{3}) = 0$

is equivalent to

$(x - \sqrt{3})(x + \sqrt{3}) = 0$

is equivalent to

$x - \sqrt{3} = 0$ or $x + \sqrt{3} = 0$

The truth set is $\{\sqrt{3}, -\sqrt{3}\}$.

(b) The truth set is $\{0, \sqrt{2}, -\sqrt{2}\}$

(c) The truth set is $\{0\}$

3. (a) $x^2 + 4x - 1 = (x^2 + 4x + 4) - 1 - 4$

$= (x + 2)^2 - (\sqrt{5})^2$

$= (x + 2 - \sqrt{5})(x + 2 + \sqrt{5})$

(b) $x^2 + 4x + 2 = (x^2 + 4x + 4) + 2 - 4$

$= (x + 2)^2 - (\sqrt{2})^2$

$= (x + 2 - \sqrt{2})(x + 2 + \sqrt{2})$

(c) $x^2 + 4x + 3 = (x + 3)(x + 1)$

(d) $x^2 - 6x + 6 = (x^2 - 6x + 9) + 6 - 9$

$= (x - 3)^2 - (\sqrt{3})^2$

$= (x - 3 - \sqrt{3})(x - 3 + \sqrt{3})$

(e) $y^2 - 5 = (y - \sqrt{5})(y + \sqrt{5})$

(f) $(z - 6 - \sqrt{2})(z - 6 + \sqrt{2})$

(g) $(s - 5 - 2\sqrt{6})(s - 5 + 2\sqrt{6})$

(h) $2(x - 2 - \sqrt{5})(x - 2 + \sqrt{5})$

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4. (a) $y^2 - 4y + 2 = 0$

$(y^2 - 4y + 4) + 2 - 4 = 0$

$(y - 2)^2 - (\sqrt{2})^2 = 0$

$(y - 2 - \sqrt{2})(y - 2 + \sqrt{2}) = 0$

$y - 2 - \sqrt{2} = 0$ or $y - 2 + \sqrt{2} = 0$

$y = 2 + \sqrt{2}$ or $y = 2 - \sqrt{2}$

The solutions are $2 + \sqrt{2}$ and $2 - \sqrt{2}$.

(b) The solutions are $3 + \sqrt{3}$ and $3 - \sqrt{3}$.

(c) The solutions are $5 + \sqrt{26}$ and $5 - \sqrt{26}$.

(d) There are no solutions.

*5. (a) 49 (d) $\frac{1}{25}$

(b) $\frac{9}{4}$ (e) $\frac{9}{64}$

(c) $\frac{1}{4}$

*6. (a) $(a + \frac{3}{2} - \frac{\sqrt{5}}{2})(a + \frac{3}{2} + \frac{\sqrt{5}}{2})$

(b) $(y + \frac{1}{2} - \frac{\sqrt{13}}{2})(y + \frac{1}{2} + \frac{\sqrt{13}}{2})$

(c) $(x - \frac{5}{2} - \frac{\sqrt{33}}{2})(x - \frac{5}{2} + \frac{\sqrt{33}}{2})$

(d) not factorable over the real numbers

*7. (a) The truth set is $\{-\frac{3}{2} - \frac{\sqrt{5}}{2}, -\frac{3}{2} + \frac{\sqrt{5}}{2}\}$

(b) The solutions are $\frac{7}{2} + \frac{\sqrt{61}}{2}$ and $\frac{7}{2} - \frac{\sqrt{61}}{2}$.

(c) The truth set is $\{\frac{5}{3}, -\frac{1}{3}\}$

(d) The solutions are -1 and $\frac{1}{2}$.

*8. (a) $2x^2 - 12x - 5 = 2(x^2 - 6x - \frac{5}{2})$

$$= 2(x^2 - 6x + 9 - 9 - \frac{5}{2})$$

$$= 2\left((x - 3)^2 - \frac{23}{2}\right)$$

$$= 2\left(x - 3 - \sqrt{\frac{23}{2}}\right)\left(x - 3 + \sqrt{\frac{23}{2}}\right)$$

(b) $3(y + \frac{1}{3} - \frac{\sqrt{7}}{3})(y + \frac{1}{3} + \frac{\sqrt{7}}{3})$

(c) $5(a - \frac{1}{10} - \frac{\sqrt{21}}{10})(a - \frac{1}{10} + \frac{\sqrt{21}}{10})$

*9. (a) $\{3 + \sqrt{\frac{23}{2}}, 3 - \sqrt{\frac{23}{2}}\}$

(b) $\{1 + \sqrt{\frac{7}{3}}, 1 - \sqrt{\frac{7}{3}}\}$

(c) $\{-\frac{5}{6} + \frac{\sqrt{13}}{6}, -\frac{5}{6} - \frac{\sqrt{13}}{6}\}$

12-7 The Algebra of Rational Expressions.

As the title of this section indicates, we hope that the student has begun to feel that, in working with expressions such as polynomials, he is dealing with a system. It may be natural to bring this point out in class if the opportunity should arise, although it is probably not a good idea to make an issue of it at this time. The better students, at least, should already be aware that our work with expressions is based on operations of addition and multiplication which have some of the same properties as the corresponding operations for real numbers. Since we have emphasized many times that these operations and their properties are what give the real number system its structure, these students should be about ready to think of the system of expressions in this more sophisticated way.

The analogy between rational expressions and rational numbers and between polynomials and integers should be emphasized.

Notice that zero (as well as 5, for example) is a rational expression. In fact, zero can even be thought of as a polynomial over the integers. If A and B are rational expressions, then $A + B$, $A - B$, AB are obviously rational expressions. Also, $\frac{A}{B}$ is a rational expression if B cannot be written as the zero expression. However, there may be restrictions on the domains of the variables involved in B in order to avoid division by zero. As an example of an expression which can be written as the zero expression, we have the expression

$$(x + y)(x - y) + y^2 - x^2$$

Since

$$(x + y)(x - y) + y^2 - x^2 = 0$$

for all values of the variables x and y, an expression such as

$$\frac{x + y}{(x + y)(x - y) + y^2 - x^2}$$

is not a numeral for any values of the variables and therefore cannot be admitted as a rational expression even with the convention of restricting the domain of variables. The expressions which can be written as the zero expression are precisely those expressions which represent the number zero for all values of the variables. In other words, they are "equal to zero" in the sense of the definition mentioned in the introductory comments on this chapter.

Answers to Problem Set 12-7; page 354:

1. $\frac{3x - 3}{x^2 - 1} = \frac{3}{x + 1}$ if $x \neq 1$ and $x \neq -1$

2. x^2 if $y \neq 1$

3. $\frac{x + 2}{x + 1}$ if $x \neq 6$, $x \neq -1$

4. $\frac{b}{1 + b}$ if $b \neq 1$, $b \neq -1$, $a \neq 0$

5. $\frac{(x + 3)(x + 1)}{2x}$ if $x \neq 3$, $x \neq -1$, $x \neq 0$

6. 1 if $x \neq 1$, $x \neq -1$

12-8. Simplification of Sums of Rational Expressions.

Answers to Problem Set 12-8; pages 357-358:

1. $$\frac{3}{x^2} - \frac{2}{5x} = \frac{3}{x^2} \cdot \frac{5}{5} - \frac{2}{5x} \cdot \frac{x}{x}$$
$$= \frac{15}{5x^2} - \frac{2x}{5x^2} = \frac{15 - 2x}{5x^2}$$

2. $$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a} \cdot \frac{bc}{bc} + \frac{1}{b} \cdot \frac{ac}{ac} + \frac{1}{c} \cdot \frac{ab}{ab}$$
$$= \frac{bc}{abc} + \frac{ac}{abc} + \frac{ab}{abc} = \frac{bc + ac + ab}{abc}$$

3. $$\frac{1}{a^2} - \frac{1}{2a} - 2 = \frac{1}{a^2} \cdot \frac{2}{2} - \frac{1}{2a} \cdot \frac{a}{a} - 2 \cdot \frac{2a^2}{2a^2}$$
$$= \frac{2}{2a^2} - \frac{a}{2a^2} - \frac{4a^2}{2a^2} = \frac{2 - a - 4a^2}{2a^2}$$

4. $$\frac{5}{x - 1} + 1 = \frac{5}{x - 1} + \frac{x - 1}{x - 1} = \frac{5 + x - 1}{x - 1} = \frac{4 + x}{x - 1}$$

5. $\frac{3}{m - 1} + \frac{2}{m - 2} = \frac{3}{m - 1} \cdot \frac{m - 2}{m - 2} + \frac{2}{m - 2} \cdot \frac{m - 1}{m - 1}$

$= \frac{3m - 6 + 2m - 2}{(m - 1)(m - 2)} = \frac{5m - 8}{(m - 1)(m - 2)}$

6. $\frac{x}{x + 5} - \frac{x}{x - 3} = \frac{x}{x + 5} \cdot \frac{x - 3}{x - 3} - \frac{x}{x - 3} \cdot \frac{x + 5}{x + 5}$

$= \frac{x^2 - 3x - x^2 - 5x}{(x + 5)(x - 3)} = - \frac{8x}{(x + 5)(x - 3)}$

7. $\frac{5m - n}{(m - n)n}$

8. $\frac{x^2 - 2xy - y^2}{(x - y)(x + y)}$

9. $\frac{5}{a - b}$

10. $\frac{12x - 21}{x^2 - 9}$

11. $\frac{-a + 3b}{(a - b)^2}$

12. $\frac{7a - 7b + 6}{(a - b)^2}$

13. $\frac{9 - 5x}{3x(x + 2)}$

14. $\frac{6a - 10}{a(a - 5)(a + 1)}$

15. $\frac{8x - 1}{(x - 2)^2 (x + 3)}$

16. $\frac{y^2 - 3y + 10}{2y^2}$

17. $\frac{9}{a(a + 3)}$

18. $\frac{b + 2}{2(b - 5)}$

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19. $\frac{5 + 2x}{x(x - 1)}$

20. $\frac{11a + 65}{6(a - 5)(a + 5)}$

21. $x - y$

22. $\frac{ab}{b - a}$

23. $\frac{x - 3}{3}$

24. The set is closed under these operations. We hope that the student is beginning to appreciate that he is dealing with another system.

12-9. Division of Polynomials.

The fundamental idea in this section is represented by the following property of the system of polynomials:

> Let N and D be polynomials with D different from zero. Then there exist polynomials Q and R with R of lower degree than D such that $N = QD + R$.

This property is analagous to the following property of the system of integers:

> Let n and d be positive integers with d different from 0. Then there exist positive integers q and r with r less than d such that $n = qd + r$.

The similarity of these properties accounts further for the parallel between polynomials and integers. Just as the division process in arithmetic is a systematic procedure for obtaining the integers q and r, the division process for polynomials is simply a systematic procedure for obtaining the polynomials Q and R.

The technique of division should not be allowed to obscure the idea behind division. Of mathematical importance here are the structure properties of the system of polynomials which are implied by the existence of Q and R. This is another case in which a standard technique in elementary algebra must pay its way by carrying along some important mathematical ideas.

Answers to Problem Set 12-9a; page 362:

1. (a) $-12a^2 - 7a + 12$
   (b) $-2x^3 - 2x^2 - 7x + 8$
   (c) $-2y^2 + 4y - 5$
2. (a) $13a - 20$
   (b) $- 11x^2 - 6x - 6$
   (c) $2y^2 + 11y - 16$
   (d) 9

Answers to Problem Set 12-9b; page 363:

1.
$$\begin{array}{r|l} \underline{x - 2} & 2x^2 - 4x + 3 = 2x(x - 2) + 3 \\ & \underline{2x^2 - 4x \qquad\quad} \\ & \qquad\quad 3 \end{array}$$

Thus $2x^2 - 4x + 3 = 2x(x - 2) + 3$

and $\frac{2x^2 - 4x + 3}{x - 2} = 2x + \frac{3}{x - 2}$

2. $4x^2 - 4x - 15 = (2x - 5)(2x + 3)$

and $\frac{4x^2 - 4x - 15}{2x + 3} = 2x - 5$

3. $2x^3 - 5x^2 - 8x + 10 = (x^2 - 4x + 2)(2x + 3) + 4$

and $\frac{2x^3 - 5x^2 - 8x + 10}{2x + 3} = x^2 - 4x + 2 + \frac{4}{2x + 3}$

4. $2x^3 - 2x^2 + 5 = (2x^2 + 10x + 60)(x - 6) + 365$

and $\dfrac{2x^3 - 2x^2 + 5}{x - 6} = 2x^2 + 10x + 60 + \dfrac{365}{x - 6}$

5. $2x^5 + x^3 - 5x^2 + 2 = (2x^4 + 2x^3 + 3x^2 - 2x - 2)(x - 1)$

and $\dfrac{2x^5 + x^3 - 5x^2 + 2}{x - 1} = 2x^4 + 2x^3 + 3x^2 - 2x - 2$

6. $3x^3 - 2x^2 + 14x + 5 = (x^2 - x + 5)(3x + 1)$

and $\dfrac{3x^3 - 2x^2 + 14x + 5}{3x + 1} = x^2 - x + 5$

Answers to Problem Set 12-9c; pages 364-365:

1.
$$\begin{array}{rll}
x - 3 \,\big| & x^3 - 3x^2 + 7x - 1 & \big|\, x^2 + 7 \\
& \underline{x^3 - 3x^2} & \\
& \qquad\qquad 7x - 1 & \\
& \qquad\qquad \underline{7x - 21} & \\
& \qquad\qquad\qquad 20 &
\end{array}$$

Check: $(x^2 + 7)(x - 3) + 20 = x^3 - 3x^2 + 7x - 1$

Therefore, $\dfrac{x^3 - 3x^2 + 7x - 1}{x - 3} = x^2 + 7 + \dfrac{20}{x - 3}$

2. $x + 3 + \dfrac{30}{x - 5}$

3. $x^3 - 3x^2 + \dfrac{-1}{x + 3}$

4. $5x^2 - 10x + 9 + \dfrac{-11}{x + 2}$

5. $2x - 5$

6. $2x^2 + x - 1 + \dfrac{2}{3x - 2}$

7. $x^3 + x^2 + x + 1$

8. $x^3 - x^2 + x - 1 + \frac{2}{x+1}$

9. $x^4 - x^3 + x^2 - x + 1$

10. $\frac{1}{3}x + \frac{2}{3} + \frac{1}{3x - 15}$ note: the quotient is a polynomial over the rational numbers.

11. $\frac{1}{2}x^2 - \frac{1}{4}x + \frac{5}{8} - \frac{\frac{13}{8}}{2x+1}$

12. $\frac{3}{2}x + \frac{1}{4} - \frac{\frac{1}{4}}{2x+1}$

13. $N = QD + R$

if $R = 0$ then $N = QD$ and D is a factor of N.

Thus if N is divided by D and there is no remainder, then D is a factor of N.

$$\begin{array}{r|l|l}
x + 3 & 2x^4 + 2x^3 - 7x^2 + 14x - 3 & 2x^3 - 4x^2 + 5x - 1 \\
\hline
 & \underline{2x^4 + 6x^3} & \\
 & \quad - 4x^3 - 7x^2 & \\
 & \quad \underline{- 4x^3 - 12x^2} & \\
 & \qquad\qquad 5x^2 + 14x & \\
 & \qquad\qquad \underline{5x^2 + 15x} & \\
 & \qquad\qquad\qquad -x - 3 & \\
 & \qquad\qquad\qquad \underline{-x - 3} &
\end{array}$$

Thus, $2x^4 + 2x^3 - 7x^2 + 14x - 3 = (2x^3 - 4x^2 + 5x - 1)(x + 3)$

Answers to Problem Set 12-9d; pages 366-367:

1. (a) $2x + 2 + \frac{9}{x - 3}$

   (b) $2x - 5 + \frac{2}{2x + 3}$

   (c) $x^2 - 4x + 2 + \frac{3}{2x + 3}$

   (d) $x^2 + \frac{8}{3}x - \frac{7}{9} + \frac{\frac{29}{9}}{3x - 1}$

   (e) $x^2 - 2x + 5 + \frac{-12}{x + 2}$

   (f) $2x - 1 + \frac{10x + 2}{x^2 - 3}$

   (g) $x + \frac{8x - 1}{x^2 - 2x - 1}$

   (h) $3x^2 + 4x + 10 + \frac{16x + 33}{x^2 - 4}$

   (i) $x - \frac{1}{5} + \frac{-\frac{2}{5}x^2 - x + 4}{5x^3 - 2x^2}$

   (j) $3x + \frac{x^2 - x + 2}{x^3 - x}$

2. (a) $3x^5 - 5x^4 + 1$

   (b) $x^6 - x^3 + 1$

   (c) $2x^2 + 2x - 1$

   (d) $(2x^4 + x - 5)(2x^4 - 5) + \frac{x^2 - 5x}{2x^4 + x - 5}$

   Therefore, $2x^4 + x - 5$ is not a factor of

   $4x^8 + 2x^5 - 20x^2 - 10x + 25$.

Answers to Review Problems; pages 369-375:

1. All except (i), (j), (k) are rational expressions.

   (a), (b), (c), (d), (e), (f), (g), (q), (r), (s), (v), (w) are polynomials.

   (b), (q), (r), (v), (w), are polynomials in one variable.

   (a), (b), (c), (d), (q), (s), (v), are polynomials over the integers.

   (e), (f), (g), are polynomials over the rational numbers but not over the integers.

   (r), (w) are polynomials over the real numbers but not over the rational numbers.

2. (a) $6\sqrt{2} + 6\sqrt{3} - \frac{1}{2}\sqrt{2} - 2\sqrt{3} = \frac{11}{2}\sqrt{2} + 4\sqrt{3}$

   (b) $\sqrt{18a^4} = 3a^2\sqrt{2}$

   (c) $(x + y)\sqrt{x + y}$ for non-negative numbers $(x + y)$

3. (a) $4\sqrt{3} - 6\sqrt{2}$

   (b) $3 + 2\sqrt{6} + 2 = 5 + 2\sqrt{6}$

   (c) $x - 1$

4. (a) $(x - 24)(x + 2)$

   (b) Not factorable over the integers.

   (c) $3a^2b^3(ab^2 - 2 + 4a^2b)$

   (d) $(x - y)(x + y) - 4(x + y) = (x + y)(x - y - 4)$

   (e) $(x - y)(x + y) + 2(x - y)(x - y) - 3(x - y)(x - y)^2$

   $= (x - y)\left((x + y) + 2(x - y) - 3(x - y)^2\right)$

   $= (x - y)(x + y + 2x - 2y - 3x^2 + 6xy - 3y^2)$

   $= (x - y)(3x - y - 3x^2 + 6xy - 3y^2)$

   (f) $(3a - 2)(2a - 5)$

   (g) $(3a - 2)(2a + 5)$

[pages 369-370]

(h) $4(x - y)^3 + 8(x - y)^2 - 2(y - x)^2$ Notice $(y - x)^2 = (x - y)^2$

$= 4(x - y)^3 + 6(x - y)^2$

$= 2(x - y)^2 \left(2(x - y) + 3\right)$

$= 2(x - y)^2 (2x - 2y + 3)$

(i) $x^2 + 2ax + a^2 - bx - ba - cx - ca$

$= (x + a)^2 - (x + a)(b + c)$

$= (x + a)(x + a - b - c)$

5. (a) Positive factors of 12 are 12, 1
6, 2
4, 3

K may be 13, 8, 7

(b) $6 = 1 + 5$
$2 + 4$
$3 + 3$

K may be 5, 8, 9

(c) $\frac{6\sqrt{3}}{2} = 3\sqrt{3} = \frac{b}{2}$

$(\frac{b}{2})^2 = K$

$(3\sqrt{3})^2 = 27$

6. (a) $\frac{9b^2}{14x}$

(b) $\frac{15b^2 + 91ab - 125a^2}{175a^2b^2}$

(c) $\frac{a - 2b}{ab(a - b)}$

(d) $\frac{3(3x - 5)}{(x + 3)(x - 3)(x - 1)}$

7. (a) $x^2 - x - 2$

(b) $x^3 + 4x^2 - 4x - 1$

(c) $x^2 - x + 1 - \frac{2}{x + 1}$

(d) $x^4 + x^3 + x^2 + x + 1$

8. (a) $\{-\frac{3}{2}\}$, if $x \neq 0$

(b) $\{25\}$, if $y \neq 0$, $y \neq 5$

(c) $\{-\frac{5}{4}\}$, if $x \neq -1$, $x \neq -2$

(d) $\{2,-7\}$, if $n \neq 3$, $n \neq -3$

(e) $\{\frac{22}{7}, \frac{20}{7}\}$, if $x \neq 3$

(f) $\{9,-9\}$

(g) $\{0, \frac{2}{3}, -\frac{2}{3}\}$

(h) $|x|^2 + |x| = 12$

$$|x|^2 + |x| - 12 = 0$$

$$(|x| + 4)(|x| - 3) = 0$$

$$\begin{array}{ll} |x| + 4 = 0 & |x| - 3 = 0 \\ |x| = -4 & |x| = 3 \\ \emptyset & x = -3 \\ & x = 3 \end{array}$$

$$\begin{array}{ll} \text{If } x = -3, & \text{If } x = 3, \\ |-3|^2 + |-3| = 12 & |3|^2 + |3| = 12 \\ 9 + 3 = 12 & 9 + 3 = 12 \end{array}$$

Hence, the truth set is $\{-3,3\}$.

9. $$(n + 3)^2 - n^2 = \big((n + 3) - n)(n + 3) + n\big)$$
$$= 3\ (2n + 3)$$

10. $$\begin{array}{r|l|l} \underline{x - 3} & x^4 - 5x^3 + 6x^2 - 3 & \underline{x^3 - 2x^2} \\ & \underline{x^4 - 3x^3} & \\ & \quad - 2x^3 + 6x^2 - 3 & \\ & \quad \underline{- 2x^3 + 6x^2 \qquad} & \\ & \qquad\qquad -3 & \end{array}$$

Therefore,

$$x^4 - 5x^3 + 6x^2 - 3 = (x^3 - 2x^2)(x - 3) - 3$$

so that $x - 3$ is not a factor.

11. (a) The degree of R is less than 3.

(b) The degree of Q is 97.

12. (a) When we use equality to indicate that one expression is "written in" another form, it is always understood to mean that the equation is a true statement for all admissible values of the variables. Hence, the truth set is all real numbers.

(b) Any value of x could be used. For example, if $x = 0$, we obtain $1 = 2\cdot(-1) + R$ and hence, $R = 3$. A better value is $x = 1$ since in this case

$$2\cdot 1^4 + 1 = 2(1^3 + 1^2 + 1^1 + 1)(1 - 1) = R$$
$$3 = 2\cdot 4\cdot 0 + R$$
$$3 = R$$

The idea is that with this value of x the first term on the right hand side of the equation is automatically zero regardless of what the number $2(x^3 + x^2 + x + 1)$ is. (See the next problem.)

13. In this problem we do not know Q and it would be a great deal of trouble to find it. However, the choice of 1 for the value of x gives

$$\begin{aligned} 5\cdot1^{100} + 3\cdot1^{17} - 1 &= Q(1 - 1) + R \\ 7 &= Q\cdot0 + R \\ 7 &= R\,. \end{aligned}$$

Therefore, we obtain the value of R in spite of not knowing what number is represented by Q when $x = 1$.

14. (a) Q has degree 7.

(b) If $R = 0$, $x - 1$ is a factor of $4x^8 + n$.

(c) If $x = 1$, then

$$\begin{aligned} 4\cdot1^8 + n &= Q\cdot0 + R \\ 4 + n &= R \end{aligned}$$

Hence, if $n = -4$, then $R = 0$.

15.
$$\begin{array}{r|l|l} \underline{x + 3} & 2x^{17} - 5x^{15} + 1 & \underline{2x^{16} - 6x^{15}} \\ & \underline{2x^{17} + 6x^{16}} & \\ & \quad - 6x^{16} - 5x^{15} + 1 & \\ & \quad \underline{- 6x^{16} - 18x^{15}} & \\ & \qquad\quad 13x^{15} + 1 & \end{array}$$

Therefore,

$$2x^{17} - 5x^{15} + 1 = (2x^{16} - 6x^{15})(x + 3) + (13x^{15} + 1)$$

16. Theorem. If a and b are distinct positive real numbers, then

$$\frac{a + b}{2} > \sqrt{ab}$$

Proof: If $a + b - 2\sqrt{ab} > 0$

then $a + b - 2\sqrt{ab} + 2\sqrt{ab} > 0 + 2\sqrt{ab}$
(addition property of order)

or $a + b > 2\sqrt{ab}$.

Hence, $\frac{a + b}{2} > \sqrt{ab}$ (multiplication property of order)

Therefore, we have only to prove that,

$$a + b - 2\sqrt{ab} > 0.$$

Observe that

$$a + b - 2\sqrt{ab} = a - 2\sqrt{a}\sqrt{b} + b$$
$$= (\sqrt{a} - \sqrt{b})^2.$$

Since, $a \neq b$, also $\sqrt{a} \neq \sqrt{b}$ and thus, $\sqrt{a} - \sqrt{b} \neq 0$. Since, the square of any non-zero real number is positive, it follows that $a + b - 2\sqrt{ab} > 0$.

17. If x is the number of minutes until they meet, then $x \cdot \frac{1}{30}$ or $\frac{x}{30}$ is the part of the whole job done by one boy, and $x \cdot \frac{1}{45}$ or $\frac{x}{45}$ is the part of the whole job done by the other boy. After x minutes have elapsed, the two fractions must total 1. Then, $\frac{x}{30} + \frac{x}{45} = 1$ and $x = 18$.

18. If n is the number of pounds of candy selling for \$1.00 per pound that are to be used in the mixture, then $40 - n$ is the number of pounds of candy selling for \$1.40 per pound that are to be used in the mixture. Then $(n)(100)$ is the value in cents of the less expensive candy in the mixture and $(40 - n)(140)$ is the value in cents of the more expensive candy in the mixture and $(40)(110)$ is the total value in cents of the mixture. Then $100n + (140)(40 - n) = (40)(110)$ and $n = 30$, $40 - n = 10$. Hence, 30 pounds of \$1.00 per pound candy were included in the mixture, and 10 pounds of \$1.40 per pound candy were included in the mixture.

19. If x is the number of gallons of mixture removed, then the amount of water at beginning minus water removed plus the water added equals water at the finish.

$$.85(100) - .85x + x = .90(100)$$

and $x = 33\frac{1}{3}$.

Thus, $33\frac{1}{3}$ gallons of mixture were removed. An equation based on the amount of salt in the solution is

$$(.15)(100) - (.15)(x) = .10(100)$$

where x again is the number of gallons of mixture removed.

20. If r is the rate of the train, then $10r$ is the rate of the jet. In 8 hours the train travels $8r$ miles and in one hour the jet will go $10r$ miles. Then,

$10r = 8r + 120.$

$r = 60$ the rate of the train in miles per hour.

$10r = 600$ the rate of the jet in miles per hour.

21. If r is the rate of one train, the $\frac{2}{3}r$ is the rate of the second. In 3 hours and 12 minutes or $3\frac{1}{5}$ hours

$$\frac{16}{5}r + \frac{16}{5}\left(\frac{2}{3}r\right) = 160.$$

$r = 30$ miles per hour.

$\frac{2}{3}r = 20$ miles per hour.

22. $\frac{300}{30} = 10$ hours one way

$\frac{300}{20} = 15$ hours returning

Since the average rate for the whole trip must involve total distance and total time, the average rate is

$\frac{600}{25}$ or 24 miles per hour.

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23. If d is the distance in miles one way $(d > 0)$ and the rate is r miles per hour, the time one way is $\frac{d}{r}$ hours. On the return, if the rate is q miles per hour, the time is $\frac{d}{q}$ hours. The total distance, 2d miles, divided by the total time, $\frac{d}{r} + \frac{d}{q}$ hours, will be

$$\frac{2d}{\frac{d}{r} + \frac{d}{q}} = \frac{2d}{\frac{d}{r} + \frac{d}{q}} \cdot \frac{rq}{rq} = \frac{2drq}{dq + dr}$$

$$= \frac{2drq}{d(q + r)} = \frac{2rq}{q + r} \cdot \frac{d}{d}$$

$$= \frac{2rq}{q + r} \text{ miles per hour}, \quad q \neq 0, \quad r \neq 0.$$

applying this to Problem 22.

$$\frac{2(30)(20)}{30 + 20} = \frac{1200}{50}$$

$$= 24 \text{ miles per hour.}$$

The student should observe that the distance traveled does not affect the average rate.

24. If x is the first integer then $x + 1$ is its successor and the reciprocals are $\frac{1}{x}$ and $\frac{1}{x + 1}$ respectively.

Hence, $\frac{1}{x} + \frac{1}{x + 1} = \frac{27}{182}$, if $x \neq 0$, $x \neq -1$.

$$182(x + 1) + 182(x) = 27x(x + 1)$$

$$(27x + 14)(x - 13) = 0$$

$$x - 13 = 0$$

$$x = 13$$

$$27x + 14 = 0$$

This equation has no solution among the integers.

If $x = 13$, then

$$\frac{1}{13} + \frac{1}{14} = \frac{27}{182} \text{ or } \frac{14}{182} + \frac{13}{182} = \frac{27}{182} .$$

Thus, the truth set of the sentence is $\{13\}$ and the required integers are 13 and 14.

25. $\dfrac{\frac{x+3}{x} + \frac{x-3}{x}}{2} = \dfrac{\frac{2x}{x}}{2} = \dfrac{2}{2} = 1$, if $x \neq 0$.

26. If x is the number, then

$$x^2 = 91 + 6x.$$

$$x^2 - 6x - 91 = 0$$

$$(x - 13)(x + 7) = 0$$

$x - 13 = 0$ $\quad$ $x + 7 = 0$

$x = 13$ $\quad$ $x = -7$

If $x = 13$, $\quad$ If $x = -7$

$13^2 = 91 + 6(13)$, $\quad$ $(-7)^2 = 91 + 6(-7)$,

$169 = 91 + 78$ $\quad$ $49 = 91 - 42$

Thus, the truth set is $\{13, -7\}$.

27. If n is the number of mph for the faster car then $n - 4$ is the number of mph for the second. Then $\frac{360}{n}$ is the number of hours during which the faster travels, and $\frac{360}{n-4}$ is the number of hours during which the slower travels. Hence,

$$\frac{360}{n} + 1 = \frac{360}{n-4}, \text{ if } n \neq 0, \ n \neq 4, \ n > 0$$

$$360(n - 4) + n(n - 4) = 360n$$

$$360n - 1440 + n^2 - 4n = 360n$$

$$n^2 - 4n - 1440 = 0$$

$$(n + 36)(n - 40) = 0$$

$n + 36 = 0$

This equation has no solution among the positive numbers.

$n - 40 = 0$

$n = 40$

If $n = 40$,

$$\frac{360}{40} + 1 = \frac{360}{36}$$

$$9 + 1 = 10$$

Hence, the positive number of the truth set is 40, and the rates of speed are 40 m.p.h. and 36 m.p.h.

28. If the width of the strip is w feet then the number of feet in the length of the rug is $20 - 2w$, and the number of feet in the width of the rug is $14 - 2w$. Hence, two names for the area of the rug are available, and appear as sides of the equation:

$$(20 - 2w)(14 - 2w) = (24)(9), \quad 0 < w < 7$$

$$280 - 68w + 4w^2 = 216$$

$$4w^2 - 68w + 64 = 0$$

$$w^2 - 17w + 16 = 0$$

$$(w - 16)(w - 1) = 0$$

$$w = 1$$

$w - 16 = 0$ has no solution such that $w < 7$

If $w = 1$

$$(20 - 2)(14 - 2) = (24)(9)$$

$$(18)(12) = 216$$

Thus, the truth set of the original equation is $\{1\}$. Hence, the width of the strip is 1 foot.

29. If x is the number of units in the length of the smaller leg, then $2x + 2$ is the number of units in the longer leg. Hence, by the Pythagorean relationship,

$$x^2 + (2x + 2)^2 = 13^2, \quad 0 < x < 13.$$

$$x^2 + 4x^2 + 8x + 4 = 169$$

$$5x^2 + 8x - 165 = 0$$

$$(5x + 33)(x - 5) = 0$$

$$x = 5$$

$5x + 33 = 0$ has no positive solution.

$$\text{If} \qquad x = 5,$$

$$5^2 + \big((2)(5) + 2\big) = 13^2$$

$$169 = 169$$

Thus, the truth set is $\{5\}$, and the shorter leg is 5 units in length, and the longer leg, 12 units.

30. $\sqrt[3]{\pi^3} = \pi$, which is irrational.

$\sqrt{.4} = \sqrt{\frac{4}{10}} = \frac{2\sqrt{10}}{10} = \frac{\sqrt{10}}{5}$, which is irrational.

$\sqrt[3]{.0008} = \sqrt[3]{\frac{8}{10000}} = \sqrt[3]{\frac{800}{1000000}} = \frac{\sqrt[3]{800}}{100} = \frac{\sqrt[3]{100}}{50}$, which is irrational.

$(\sqrt[3]{-1})(\sqrt{.16}) = (-1)(.4) = -.4$ which is rational.

31. If the two-digit number is $10t + u$, the sum of its digits is $t + u$, and

$$\frac{10t + u}{t + u} = 4 + \frac{3}{t + u}$$

$$10t + u = 4t + 4u + 3$$

$$6t - 3u = 3$$

$$2t - u = 1$$

$u = 2t - 1$ and u is a positive integer ≤ 9.

If $t = 1$, then $u = 1$, and the number is 11;

if $t = 2$, then $u = 3$, and the number is 23;

if $t = 3$, then $u = 5$, and the number is 35;

if $t = 4$, then $u = 7$, and the number is 47;

if $t = 5$, then $u = 9$, and the number is 59.

If $t = 1$, $u = 1$, then $\frac{10t + u}{t + u}$ is $\frac{11}{2} = 4 + \frac{3}{2}$. The pair of values, $t = 1$, $u = 1$, should not be allowed since the numberator, 3, of the remainder is greater than the denominator, 2.

If $t = 2$, and $u = 3$, $\frac{23}{5} = 4 + \frac{3}{5}$;

if $t = 3$, and $u = 5$, $\frac{35}{8} = 4 + \frac{3}{8}$;

if $t = 4$, and $u = 7$, $\frac{47}{11} = 4 + \frac{3}{11}$;

if $t = 5$, and $u = 9$, $\frac{59}{14} = 4 + \frac{3}{14}$.

Hence, the solutions are 23, 35, 47, 59.

32. $\frac{8}{5}$

33.
$$|x - 5|^2 \geq 9$$
$$|x - 5| \cdot |x - 5| \geq 3 \cdot 3$$
$$|x - 5| \geq 3$$
$$x - 5 \geq 3 \quad \text{or} \quad x - 5 \leq -3$$
$$x \geq 8 \quad \text{or} \quad x \leq 2$$

The truth set is the set of all x such that $x \geq 8$ or $x \leq 2$. The formal methods for solving inequalities such as this are not yet available, so the student will have to move toward the solution by careful trial of numerous possible members of the set, or, if he is seeking a less haphazard approach, he may make the plausible assumption that if a and b are positive numbers and $a^2 > b^2$, then $a > b$.

34. (a) While the small hand travels over a number of minute markings, x, the large hand travels over $12x$ of these units. Since the hour hand is at 3 o'clock position, it has a 15-unit "head-start" over the minute hand at the time 3:00. Thus

$$12x = x + 15.$$
$$11x = 15,$$
$$x = \frac{15}{11}.$$

If $x = \frac{15}{11}$, then

$$12\left(\frac{15}{11}\right) = \frac{15}{11} + 15,$$
$$\frac{180}{11} = \frac{180}{11}.$$

Thus, the truth set of the equation is $\{\frac{15}{11}\}$ and the time when the hands are together is $16\frac{4}{11}$ minutes after 3 o'clock.

(b) In part (a) both hands came to the same minute division; in part (b) the minute hand is to come to a reading 30 units ahead of the hour hand. Hence, an equation for part b is

$$12x = (x + 15) + 30 \quad \text{and} \quad x = 4\frac{1}{11}$$

And the hands will be opposite each other at $49\frac{1}{11}$ minutes after 3 o'clock.

35. If the number of steers is s and number of cows is c, then,

$$25s + 26c = 1000$$

$$25s = 1000 - 26c$$

$$s = \frac{1000 - 26c}{25}$$

$$s = 40 - \frac{26c}{25}$$

If s and c are positive integers then 26c must be divisible by 25. This is true when c = 25, 50, 75,..., a multiple of 25, because 26 and 25 are relatively prime to each other.

If $c = 25$, $\frac{26c}{25} = 26$ and $s = 40 - 26 = 14$.

If $c = 50$, $\frac{26c}{25} = 52$ and $s = 40 - 52 = -12$.

If $c = 75$, $\frac{26c}{25} = 78$ and $s = 40 - 78 = -38$.

It is thus apparent that if $c \geq 50$, s is a negative number. Hence, c may only be 25

$$\text{and} \quad s = 40 - 26$$

$$s = 14.$$

So he may buy 25 cows and 14 steers.

If we were to solve the original equation instead for c,

$$c = \frac{1000 - 25s}{26}$$

s would have to be chosen so as to make 1000 - 25s divisible by 26. Though this can be done, it is plainly more difficult than the other approach.

Suggested Test Items

1. Classify the following expressions by writing the identifying letter of the expression in the spaces provided. One expression may fall into more than one classification.

(a) $(x - 3)(x + 2)$

(b) $s + \frac{1}{3}t$

(c) $\sqrt{x + 1}$

(d) $3x^{-2} + x^{-1}$

(e) $\sqrt{3}x^2 + 1$

(f) $\frac{(x - 1)(x - 2)}{(x - 2)}$

(g) $|x - 1|$

rational expressions ____________________
polynomials over the real numbers ____________________
polynomials over the rational numbers ____________________
polynomials over the integers ____________________
none of the above

2. Factor over the integers, if possible.

(a) $ax^2 - ax - 6a^2$

(b) $9a^2 - 16$

(c) $x^2 - x - 20$

(d) $6x^2 - 11x - 72$

(e) $(2a - b)^2 - (a - 2b)^2$

(f) $3ab - 3b^2 + 2a^2 - 2ab$

3. Find the truth sets of the following:

(a) $x^2 + 3x = 54$

(b) $y^2 + 56 = 15y$

(c) $3a^2 + 8a = 3$

(d) $9b^2 = 11b - 36$

(e) $x^2 + 2 = 4x$

(f) $27x^2 = 42x + 49$

4. For what values of x, is a triangle having sides of $x - 7$ inches, x inches, and $x + 1$ inches, a right triangle?

5. Prove or disprove that $2x - 1$ is a factor of

$$6x^3 - 3x^2 - 2x + 1.$$

6. For each integer y show that $(y + 5)^2 - y^2$ is divisible by 5.

7. Write an expression for x in terms of a and b if $ax - a^2 = ab - bx$.

8. If $(x + 1)(x + 2)(x + 3)(x + a) = x^4 + 8x^3 + 23x^2 + 22x + 12$ where a is an integer, find a.

9. Simplify

   (a) $\dfrac{3}{x^2 - 4x - 5} + \dfrac{4}{x^2 + x}$

   (b) $\dfrac{z - 3}{2z} - \dfrac{z + 5}{z^2}$

   (c) $\dfrac{3a}{a - b} \cdot \dfrac{b - a}{6a^2}$

   (d) $\dfrac{1 + \frac{1}{x}}{2 + \frac{2}{x}}$

   (e) $\dfrac{a^2 - 11a - 26}{a^2 - 5a + 6}$

10. Explain when $\dfrac{(x - 1)(x - 2)}{(x - 2)}$ is equal to $(x - 1)$.

11. Consider the set of polynomials over the even integers. Is this set closed under addition? Is this set closed under multiplication?

12. Find the integers a, b, and c in the following

    $(bx + 2)(3x - a) = 6x^2 - 5x - c.$

13. By what polynomial would $x + 2$ be multiplied to get the polynomial

    $x^5 + 2x^4 - x^2 + 5x + 14.$

Chapter 13

TRUTH SETS OF OPEN SENTENCES

In this chapter we take a more careful look at the process of finding the truth set of a sentence. By developing a rigorous theory of equivalent equations and equivalent inequalities, we are able to determine when a new sentence has the same truth set as the original sentence without having to check in the original sentence.

Material on open sentences and equivalent sentences will be found in _Studies in Mathematics_, Volume III, pages 6.8-6.16.

13-1 Equivalent Open Sentences.

The important concept being emphasized here is an understanding of _why_ sentences are equivalent. If your students have any trouble with the technique of deciding _what_ to _do_ to a sentence to obtain a simpler sentence, you may want to point out how an indicated addition can be "undone" by adding the opposite (as in adding $(-x-7)$ in Example 1) and an indicated multiplication can be "undone" by multiplying by the reciprocal (as in multiplying by $\frac{1}{2}$ in Example 1).

Answers to Problem Set 13-1a; pages 379-381:

1. In parts (a), (b), (c), (d), (g), (h), (i), (k), (m) the sentences are equivalent.

   (a)
$$\begin{array}{ll}
2s = 12 & s = 6 \\
\frac{1}{2}(2s) = \frac{1}{2} \cdot 12 & 2s = 2 \cdot 6 \\
s = 6 & 2s = 12
\end{array}$$

(b)	$5s = 3s + 12$	$2s = 12$
	$5s - 3s = (3s + 12) - 3s$	$2s + 3s = 12 + 3s$
	$2s = 12$	$5s = 3s + 12$
(c)	$5y - 4 = 3y + 8$	$y = 6$
	$5y - 3y = 8 + 4$	$2y = 12$
	$2y = 12$	$5y - 3y = 8 + 4$
	$y = 6$	$5y - 4 = 3y + 8$
(d)	$7s - 5s = 12$	$s = 6$
	$2s = 12$	$2s = 12$
	$s = 6$	$7s - 5s = 12$

(e) Not equivalent. 2 is a member of the truth set of $x^2 = 4$, but not of $2x^2 + 4 = 10$.

(f) Not equivalent. $\frac{7}{2}$ is a member of the truth set of $3x + 9 - 2x = 7x - 12$, but not of $\frac{7}{3} = x$.

(g)	$x^2 = x - 1$	$1 = x - x^2$
	$x^2 + 1 = x$	$x^2 + 1 = x$
	$1 = x - x^2$	$x^2 = x - 1$
(h)	$\frac{y - 1}{\lvert y\rvert + 2} = 3$	$y - 1 = 3(\lvert y\rvert + 2)$
	$\frac{y - 1}{\lvert y\rvert + 2}(\lvert y\rvert + 2) = 3(\lvert y\rvert + 2)$	$(y-1)\frac{1}{\lvert y+2\rvert} = 3(\lvert y\rvert + 2)\frac{1}{\lvert y\rvert + 2}$
	$y - 1 = 3(\lvert y\rvert + 2)$	$\frac{y - 1}{\lvert y\rvert + 2} = 3$
(i)	$x^2 + 1 = 2x$	$(x - 1)^2 = 0$
	$x^2 - 2x + 1 = 0$	$x^2 - 2x + 1 = 0$
	$(x - 1)^2 = 0$	$x^2 + 1 = 2x$

[pages 379-380]

(j) Not equivalent. 1 is a member of the truth set of $x^2 - 1 = x - 1$, but not of $x + 1 = 1$.

(k) $$\frac{x^2 + 5}{x^2 + 5} = 0 \qquad\qquad x^2 + 5 = 0$$

$$\frac{x^2 + 5}{x^2 + 5}(x^2+5) = 0(x^2+5) \qquad\qquad (x^2+5)\frac{1}{x^2+5} = 0\frac{1}{x^2+5}$$

$$x^2 + 5 = 0 \qquad\qquad \frac{x^2 + 5}{x^2 + 5} = 0$$

($x^2 + 5$ is a non-zero real number for every value of x.)

(l) Not equivalent. 0 is a member of the truth set of $\frac{x^2 + 5}{x^2 + 5} = 1$, but not of $x^2 + 5 = 1$.

(m) Not equivalent -1 is a member of the truth set of $|v + 1| = 0$, but not of $v^2 + 1 = 0$.

2. The sentences are equivalent in (a), (b), (c), and (f).

3, (a) $y = 12$ (d) $s = \frac{1}{15}$

(b) $x = 20$ (e) $x = 2$

(c) $t = -1$ (f) $y = 1$

4. (a) $11t + 21 = 32$

$11t = 11$

$t = 1$

The truth set is $\{1\}$.

(b) $\frac{4}{3} - \frac{y}{5} = \frac{1}{2}$

$(\frac{4}{3} - \frac{y}{5})30 = \frac{1}{2} \cdot 30$

$40 - 6y = 15$

$40 - 15 = 6y$

$25 = 6y$

$\frac{25}{6} = y$

The truth set is $\{\frac{25}{6}\}$.

(c) $\{80\}$

(d) $\{0\}$

(e) $\{6\}$

(f) $\emptyset$

(g) $\{-5\}$

(h) $\{\frac{3}{2}\}$

(i) $\emptyset$

(j) $y^4 + y^3 + y^2 + y + 1 = y^4 - y^3 + y^2 - y + 1$

$2y^3 + 2y = 0$

$2y(y^2 + 1) = 0$

$2 = 0$ or $y = 0$ or $y^2 + 1 = 0$

Since 2 and $y^2 + 1$ can never be 0,

the truth set is $\{0\}$.

(k) $x^2 + 3x = x - \frac{x^2}{2}$

$2x^2 + 6x = 2x + x^2$

$x^2 + 4x = 0$

$x(x + 4) = 0$

$x = 0$ or $x + 4 = 0$

$x = 0$ or $x = -4$

The truth set is $\{0, -4\}$

5. Any algebraic simplification is permissible which

   (1) does not change the domain of the variable,

   (2) replaces a phrase with another phrase which is a name for the same number for all permitted values of the variable.

   Such simplifications include combining terms and factoring as in the first two examples. On the other hand $\frac{x^2 - 4}{x - 2} = 4$ and $x + 2 = 4$ are not equivalent. Their domains are not the same.

6. (a) Equivalent; combined terms in left member.

   (b) Equivalent; factored in left member.

   (c) Not equivalent; 0 is a member of the truth set of $3x^2 = 6x$, but not of $3x = 6$.

   (d) Equivalent; added a real number $(-6x)$ to both members.

   (e) Equivalent; combined terms in both numbers.

   (f) Equivalent; if $x = y$, then $y = x$.

   (g) Not equivalent; 0 is a member of the truth set of $2 = y + 2$, but not of $2 = \frac{y^2 + 2y}{y}$.

   (h) Equivalent. Applied distributive property and combined terms in left member.

Answers to Problem Set 13-1b; pages 383-385:

1.	A real number for every value of the variable.	A non-zero real number for every value of the variable.
(a)	yes	no
(b)	no	no
(c)	no	no
(d)	no	no
(e)	yes	no
(f)	yes	yes
(g)	yes	no
(h)	yes	yes
(i)	yes	yes
(j)	yes	yes
(k)	no	no
(ℓ)	no	no

In part (j) it may appear that there is no variable involved. In certain contexts, however, -3 may be considered as an expression in x such as $-3 + 0x$. If we have such a variable in mind, it is certainly true that -3 remains a non-zero real number no matter what value is assigned to the variable.

2. (a) $\frac{y}{y - 2} = 3$ and $y \neq 2$

$y = 3(y - 2)$ and $y \neq 2$

$y = 3y - 6$ and $y \neq 2$

$6 = 2y$ and $y \neq 2$

$3 = y$ and $y \neq 2$

The truth set is $\{3\}$

(b) $\frac{x}{x^2 + 1} = x$

$x = x(x^2 + 1)$

$x = x^3 + x$

$0 = x^3$

$0 = x$

The truth set is $\{0\}$

(Since, $x^2 + 1$ is a non-zero real number for all values of x, it was not necessary to restrict the domain of x.)

(c) $\frac{1}{x} + 3 = \frac{2}{x}$ and $x \neq 0$

$1 + 3x = 2$ and $x \neq 0$

$3x = 1$ and $x \neq 0$

$x = \frac{1}{3}$ and $x \neq 0$

The truth set is $\{\frac{1}{3}\}$.

(d) $\frac{1}{x - 2} + \frac{x - 3}{x - 2} = 2$ and $x \neq 2$

$1 + (x - 3) = 2(x - 2)$ and $x \neq 2$

$x - 2 = 2x - 4$ and $x \neq 2$

$2 = x$ and $x \neq 2$

The truth set is $\emptyset$.

(e) $-\frac{1}{x + 1} + 1 = \frac{x}{x + 1}$ and $x \neq -1$

$-1 + (x + 1) = x$ and $x \neq -1$

$x = x$ and $x \neq -1$

The truth set is the set of all real numbers except -1.

(f) $x(x^2 + 1) = 2x^2 + 2$

$x(x^2 + 1) = 2(x^2 + 1)$

$x = 2$

The truth set is $\{2\}$.

(We were permitted to multiply both members by $\frac{1}{x^2 + 1}$ and be sure of obtaining an equivalent sentence because $\frac{1}{x^2 + 1}$ is a non-zero real number for all values of x.)

3. If the rectangle is w inches wide, then it is $15 - w$ inches long, and

$w(15 - w) = 54$

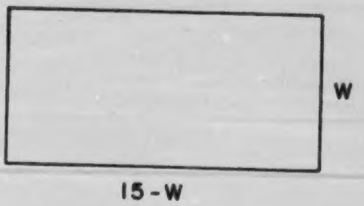
$15w - w^2 = 54$

$0 = w^2 - 15w + 54$

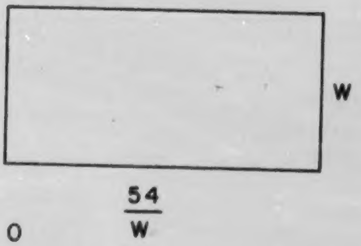
$0 = (w - 9)(w - 6)$

$w - 9 = 0$ or $w - 6 = 0$

$w = 9$ or $w = 6$



or If the rectangle is w inches wide, it is $\frac{54}{w}$ inches long, and



$2w + 2 \cdot \frac{54}{w} = 30$ and $w \neq 0$

$2w^2 + 108 = 30w$ and $w \neq 0$

$2w^2 - 30w + 108 = 0$ and $w \neq 0$

$2(w - 9)(w - 6) = 0$ and $w \neq 0$

$2 = 0$ or $w - 9 = 0$ or $w - 6 = 0$ and $w \neq 0$.

The truth set is $\{6,9\}$

The rectangle is 6 inches wide and 9 inches long.

[page 384]

(Notice that we cannot use the value $w = 9$ since we designated w as the width, or the shorter side. The length, 9 inches, comes from $15 - w$ in the first method and $\frac{54}{w}$ in the second method.)

4. If the first integer is i, then

$$i^2 + (i + 1)^2 + (i + 2)^2 = 61$$

$$i^2 + i^2 + 2i + 1 + i^2 + 4i + 4 = 61$$

$$3i^2 + 6i - 56 = 0$$

Since, the left member is not factorable over the integers, there are no three successive integers the sum of whose squares is 61.

5. If n is one of the numbers, then the other number is $8 - n$, and

$$\frac{1}{n} + \frac{1}{8 - n} = \frac{2}{3} \qquad \text{and} \quad n \neq 8, \quad n \neq 0$$

$$3(8 - n) + 3n = 2n(8 - n) \qquad \text{and} \quad n \neq 8, \quad n \neq 0$$

$$24 - 3n + 3n = 16n - 2n^2 \qquad \text{and} \quad n \neq 8, \quad n \neq 0$$

$$2n^2 - 16n + 24 = 0 \qquad \text{and} \quad n \neq 8, \quad n \neq 0$$

$$n^2 - 8n + 12 = 0 \qquad \text{and} \quad n \neq 8, \quad n \neq 0$$

$$(n - 6)(n - 2) = 0 \qquad \text{and} \quad n \neq 8, \quad n \neq 0$$

$$n - 6 = 0 \quad \text{or} \quad n - 2 = 0 \qquad \text{and} \quad n \neq 8, \quad n \neq 0$$

$$n = 6 \quad \text{or} \quad n = 2 \qquad \text{and} \quad n \neq 8, \quad n \neq 0$$

The truth set is $\{6, 2\}$

One number is 6 and the other is $8 - 6$, or 2.

6. If there were g girls, then there were $(2600 - g)$ boys,

$\frac{2600 - g}{g} = \frac{7}{6}$ and $g \neq 0$

$6(2600 - g) = 7g$ $g \neq 0$

$15600 - 6g = 7g$ $g \neq 0$

$15600 = 13g$ $g \neq 0$

$1200 = g$ $g \neq 0$

The truth set is $\{1200\}$

There were 1200 girls.

For some purposes it is convenient to know that if two numbers have the ratio $\frac{a}{b}$, the numbers may be represented as ax and bx where x is a positive number, since $\frac{ax}{bx} = \frac{a}{b}$ if $x \neq 0$ and $b \neq 0$. In this problem, then, we could say:

If there were $7x$ girls, then there were $6x$ boys, (since, $\frac{7x}{6x} = \frac{7}{6}$, $x \neq 0$), and $7x + 6x = 2600$.

This type of problem does not appear frequently enough in this course to warrant making much of this technique. You may wish to mention it.

7. $3x + 18 = y + 23$

is equivalent to

$3x + 18 + (-23) = y + 23 + (-23)$

$3x - 5 = y$

$y = 3x - 5$

Hence, the two sets of solution pairs are identical.

8. A chain of equivalent sentences:

$$4x - \frac{2}{3}y = 6$$

$$12x - 2y = 18$$

$$12x - 18 = 2y$$

$$6x - 9 = y$$

$$y = 6x - 9$$

9. $\frac{x}{2x - 5} = \frac{4}{6}$

The sides are of length 10 and 15.

10. If k quarts of weed killer are used, then 40-k quarts of water are used.

$$\frac{k}{40 - k} = \frac{3}{17}$$

There should be 6 quarts of weed-killer.

13-2. Equivalent Inequalities.

Just as for equations, the thing we must look for in establishing the fact that two inequalities are equivalent is whether the operations we perform, can be reversed to carry us back from the simpler one to the given one. If they can be reversed, we know that the truth set of the original inequality is a subset of the truth set of the new inequality and the truth set of the new one is a subset of the original one. The two truth sets are therefore identical.

Page385. As with equations, you may want to point out to your students that inequalities may be simplified if we know how to "undo" some of the indicated operations. Indicated additions can be "undone" by adding the opposite, and indicated multiplications can be "undone" by multiplying by the reciprocal.

Page 386. In Example 1 and Example 2, the truth set of the final inequality is the truth set of the original inequality because only operations yielding equivalent inequalities were used. No checking is necessary.

Answers to Problem Set 13-2; pages 387-388:

1. (a) $x + 12 < 39$

$x < 27$

The truth set is the set of all real numbers less than 27.

(b) $\frac{5}{7}x < 36 - x$

$\frac{12}{7}x < 36$

$x < 36 \cdot \frac{7}{12}$

$x < 21$

The truth set is the set of all real numbers less than 21.

(c) The set of all real numbers greater than $\sqrt{2}$.

(d) The set of all real numbers less than $\sqrt{3}$.

(e) The set of all real numbers greater than 2.

(f) The set of all real numbers less than 12.

(g) The set of all real numbers.

(h) $\emptyset$

(i) The set of all real numbers.

2. (a) $1 < 4x + 1$ and $4x + 1 < 2$

$0 < 4x$ and $4x < 1$

$0 < x$ and $x < \frac{1}{4}$

The truth set is the set of all real numbers between 0 and $\frac{1}{4}$.

(b) $4t - 4 < 0$ and $1 - 3t < 0$

$4t < 4$ and $1 < 3t$

$t < 1$ and $\frac{1}{3} < t$

The truth set is the set of all real numbers between $\frac{1}{3}$ and 1.

(c) The set of all real numbers between $-\frac{1}{2}$ and $\frac{1}{2}$

(d) The set of all real numbers which are either less than $-\frac{1}{2}$ or greater than $\frac{1}{2}$.

(e) $|x - 1| < 2$

On the number line the distance between x and 1 must be less than 2. Hence,

$$1 - 2 < x < 1 + 2$$

$$-1 < x < 3$$

The set of all real numbers between - 1 and 3.

(f) $|2t| < 1$

$2\,|t| < 1$

$|t| < \frac{1}{2}$

On the number line the distance between t and the origin must be less than $\frac{1}{2}$.

The set of all real numbers between $-\frac{1}{2}$ and $\frac{1}{2}$.

(g) $|x + 2| < \frac{1}{2}$

$|x - (-2)| < \frac{1}{2}$

The set of all real numbers between $-\frac{5}{2}$ and $-\frac{3}{2}$.

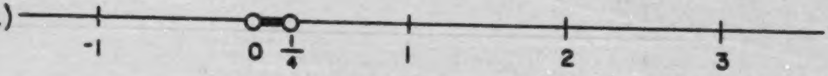
(h) $|y + 2| > 1$

$|y - (-2)| > 1$

The set of all real numbers which are either less than -3 or greater than -1.

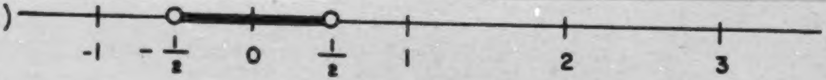
3.

(a)



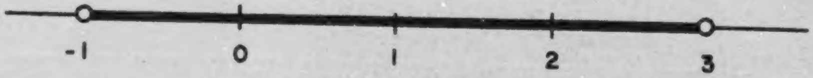
-1
0
$\frac{1}{4}$
1
2
3

(c)

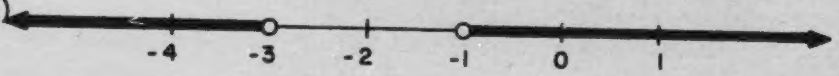


-1
$-\frac{1}{2}$
0
$\frac{1}{2}$
1
2
3

(e)



(h)



4. (c), (e) and (f) are negative real numbers for every value of x. See the note for Problem 1(j) in Problem Set 13-1b.

5. $3y - x + 7 < 0$

$3y < x - 7$

$y < \frac{1}{3}(x - 7)$

When $x = 1$, $y < \frac{1}{3}(1 - 7)$

$y < -2$

The truth set is the set of all real numbers less than -2.

$3y - x + 7 < 0$

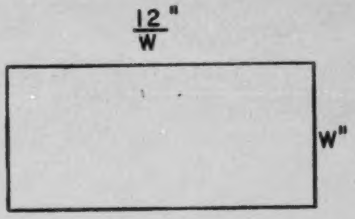
$-x < -3y - 7$

$x > 3y + 7$

When $y = -2$, $x > 3(-2) + 7$

$x > 1$

The truth set is the set of all real numbers greater than 1.



6. If the rectangle is w inches wide, it is $\frac{12}{w}$ inches long, since the area is 12 square inches. Then $\frac{12}{w} < 5$. Since, by the nature of the problem, $w > 0$,

$$12 < 5w$$

$$\frac{12}{5} < w$$

The width of the rectangle is greater than $2\frac{2}{5}$ inches.

7. If n is the negative number, then

$n < \frac{1}{n}$ and $n < 0$

$n^2 > 1$ and $n < 0$

$n < -1$

The truth set is the set of all real numbers less than -1.

At this point the students have no formal way of solving $n^2 > 1$. They can, however, go back to the method of making an intelligent guess and verifying it with the help of the number line. This is still a useful method when we do not have a better one. Later in Section 13-6 the student may look more closely at solving sentences such as $n^2 - 1 > 0$.

13-3. Equations Involving Factored Expressions.

Page 388. If $(x - 3)(x + 2) = 0$, then $x - 3 = 0$ or $x + 2 = 0$. If $x - 3 = 0$ or $x + 2 = 0$, then $(x - 3)(x + 2) = 0$. Again, the fact that this process is reversible makes it possible for us to know that

$$(x - 3)(x + 2) = 0$$

and $$x - 3 = 0 \text{ or } x + 2 = 0$$

are equivalent sentences.

If there are several factors, as in $abcd = 0$, then the equivalent sentence is

$$a = 0 \text{ or } b = 0 \text{ or } c = 0 \text{ or } d = 0.$$

The truth set of $(x + 1)(x - 3)(2x + 3)(3x - 2) = 0$ is $\{-1, 3, -\frac{3}{2}, \frac{2}{3}\}$.

Notice what can be said about an equation such as

$$3x(x + 2)(x - 3) = 0.$$

An equivalent sentence is

$$3 = 0 \text{ or } x = 0 \text{ or } x + 2 = 0 \text{ or } x - 3 = 0.$$

Since the truth set of $3 = 0$ is $\emptyset$, the truth set of the given equation is $\{0, -2, 3\}$.

Answers to Problem Set 13-3a; pages 388-389:

1. (a) $(a + 2)(a - 5) = 0$

   $a + 2 = 0$ or $a - 5 = 0$

   $a = -2$ or $a = 5$

   The truth set is $\{-2,5\}$.

   (b) $\{-3, -1, 2, 0\}$.

   (c) $\{\frac{1}{3}, -\frac{1}{2}, \frac{3}{4}\}$

2. (a) $x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x - 2 = 0$ or $x + 1 = 0$

$x = 2$ or $x = -1$

The truth set is $\{2, -1\}$.

(b) $\{11, -11\}$

(c) $\{1, -1, -3, -2\}$

(d) $\{\sqrt{5}, -\sqrt{5}, 2\sqrt{6}, -2\sqrt{6}\}$.

(e) $\{0, 5, -5\}$

(f) $\{-\frac{1}{2}, 3\}$

(g) $\{0,1\}$

(h) $\emptyset$

(i) $\{6,1\}$

(j) $\{2 + \sqrt{2}, 2 - \sqrt{2}\}$ (See Problem 3 in Problem Set 12-6).

(k) $\{-3 + \sqrt{10}, -3 - \sqrt{10}\}$.

3. We guess that 2 is a solution.

x cannot be negative or zero.

If $x > 0$ and $x < 2$

$$x^2 < 2x$$

and

$$2x < 4$$

from which $x^2 < 4$ by the transitive property of order. If $x^2 < 2x$ and $x > 0$,

$$x^3 < 2x^2$$

and if

$$x^2 < 4$$

$$2x^2 < 8$$

Hence, $x^3 < 8$ by the transitive property of order. This shows that no number less than 2 is a solution.

Similarly no number greater than 2 is a solution. Since, $2^3 = 8$, the truth set is $\{2\}$.
It should be sufficient if the student argues intuitively that if $x < 2$, $x^3 < 8$ and if $x > 2$, $x^3 > 8$.

4. $$x^4 = 1$$
$$(x^2)^2 - 1 = 0$$
$$(x^2 + 1)(x^2 - 1) = 0$$
$$(x^2 + 1)(x + 1)(x - 1) = 0$$
$$x^2 + 1 = 0 \quad \text{or} \quad x + 1 = 0 \quad \text{or} \quad x - 1 = 0$$
$$x = -1 \quad \text{or} \quad x = 1$$

The truth set is $\{-1, 1\}$.

5. $(x - 1)(x + 1)x$

6. $(x - 3)(x - 1)(x + 1) = 0$ and $|x - 2| < 2$
$(x - 3 = 0$ or $x - 1 = 0$ or $x + 1 = 0)$ and $(0 < x < 4)$
$(x = 3$ or $x = 1$ or $x = - 1)$ and $(0 < x < 4)$
The truth set is $\{3, 1\}$.

Page 390. There are times when we are tempted, or even forced, to do operations on sentences which will not necessarily give equivalent sentences.

One of the temptations is to eliminate a factor which we see in every term by multiplying by its reciprocal. If the reciprocal is not a real number---if its denominator is zero for some x, or if a square root is present---this may cause trouble. In the example given, the original equation has the truth set $\{7, 1, -1\}$, while the equation obtained by eliminating the factor $x^2 - 1$ has the truth set $\{7\}$.

Page 390. The generalization given is of interest and the students should be encouraged to follow carefully the steps. In practice, however, it is probably best to proceed as was done in the example above, not just to apply the conclusion of the generalization.

Answers to Problem Set 13-3b; page 391:

1. (a) $x(2x - 5) = 7x$

$x(2x - 5) - 7x = 0$

$x(2x - 5 - 7) = 0$

$x(2x - 12) = 0$

$x = 0$ or $2x - 12 = 0$

$x = 0$ or $x = 6$

The truth set is $\{0,6\}$

(b) $\{-3,2,-2\}$

(c) $\{2,-3\}$

(d) $\{2,-2,0\}$

(e) $\{5,3\}$ (Notice that each member of the original equation can be factored.)

2. No. 1 is a member of the truth set of

$(x - 1)x^2 = (x - 1)3$,

but 1 is not a member of the truth set of $x^2 = 3$. Since $x - 1$ is 0 when $x = 1$, we would not expect the two sentences necessarily to be equivalent.

3. The truth set of $t^2 = 1$ is $\{1,-1\}$.

The truth set of $(t + 1)t^2 = (t + 1)\cdot 1$ is $\{1,-1\}$.

In Problem 2 the truth set was enlarged by multiplying by $(x - 1)$.

In Problem 3 the truth set was not enlarged by multiplying by $(t + 1)$ because (-1), the number which makes $t + 1$ equal to 0, is already a member of the truth set of $t^2 = 1$.

13-4. Fractional Equations.

Usually in order to simplify a fractional equation we multiply by an expression that is a product of factors of the denominators in the equation. This expression may not be a non-zero real number and we have been warned that this may not give an equivalent equation. We find, however, that we can avoid trouble if we are careful to exclude the values of the variable which make the multiplier zero; so we must be careful to exclude values of the variable which make any one of the denominators zero. Thus in $\frac{1}{x} = \frac{1}{1 - x}$ we require that $x \neq 0$ and $x \neq 1$.

Page 392. The sentence $\frac{x + 1}{x - 2} = 0$ is equivalent to the sentence "$x + 1 = 0$ and $x - 2 \neq 0$", or to "$x = -1$ and $x \neq 2$". The truth set of the last sentence, and therefore of the first sentence, is $\{-1\}$.

The "suitable polynomial" is $x(1 - x)$.

Page 393. The example which comes out equivalent to "$x \neq 2$ and $x = 2$" has $\emptyset$, of course, as its truth set.

In Problem Set 13-4, Problems 10 and 11 are of particular interest and importance because they illustrate some of the unusual things which can happen. They show why it is necessary to keep in mind the domain of the variable.

Answers to Problem Set 13-4; pages 393-394:

1. $(\frac{2}{x} - \frac{3}{x})x = 10x$ and $x \neq 0$

   $2 - 3 = 10x$ and $x \neq 0$

   $-\frac{1}{10} = x$ and $x \neq 0$

   Solution: $-\frac{1}{10}$.

2. $(\frac{x}{2} - \frac{x}{3})6 = 10 \cdot 6$

   $3x - 2x = 60$

   $x = 60$

   Soltuion: 60.

3. $(x + \frac{1}{x})x = 2x$ and $x \neq 0$

   $x^2 + 1 = 2x$ and $x \neq 0$

   $x^2 - 2x + 1 = 0$ and $x \neq 0$

   $(x - 1)^2 = 0$ and $x \neq 0$

   $(x - 1 = 0$ or $x - 1 = 0)$ and $(x \neq 0)$

   $x = 1$ and $x \neq 0$

   The truth set is $\{1\}$.

4. $\{2,-1\}$

5. $\{\frac{3}{2}\}$

6. $\{-\frac{3}{32}\}$

7. $(\frac{1}{y} - \frac{1}{y - 4})y(y - 4) = 1 \cdot y(y - 4)$ and $y \neq 0$ and $y \neq 4$

   $(y - 4) - y = y^2 - 4y$ and $y \neq 0$ and $y \neq 4$

   $0 = y^2 - 4y + 4$ and $y \neq 0$ and $y \neq 4$

   $0 = (y - 2)(y - 2)$ and $y \neq 0$ and $y \neq 4$

   $y = 2$ and $y \neq 0$ and $y \neq 4$

   The truth set is $\{2\}$.

[page 393]

8. $\emptyset$

9. $\{-\frac{1}{3}, -3\}$

10. $\left(\frac{-2}{x-2} + \frac{x}{x-2}\right)(x-2) = 1\cdot(x-2)$ and $x \neq 2$

$-2 + x = x - 2$ and $x \neq 2$

$-2 = -2$ and $x \neq 2$

The truth set consists of all numbers in the truth set of $-2 = -2$ that are not 2. The truth set of $-2 = -2$ is the set of all real numbers. The desired truth set consists of all real numbers except 2.

11. $\left(\frac{x}{x+1}\right)(x^2 - 1)(x + 1) = 0$ and $x + 1 \neq 0$

$x(x^2 - 1) = 0$ and $x + 1 \neq 0$

$x(x - 1)(x + 1) = 0$ and $x \neq -1$

$(x = 0$ or $x = 1$ or $x = -1)$ and $x \neq -1$

The truth set is $\{0,1\}$.

12. $\{0\}$

13. $\emptyset$

14. $\{0\}$

*15. $\{-\frac{1}{3}\}$

*16. $\{1 + \sqrt{2}, 1 - \sqrt{2}\}$ (The method of Problem 3 in Problem Set 12-6 is needed here.)

17. If n is the number,

$n + \frac{1}{n} = -2$ and $n \neq 0$

$n^2 + 1 = -2n$ $\quad n \neq 0$

$n^2 + 2n + 1 = 0$ $\quad n \neq 0$

$(n + 1)^2 = 0$ $\quad n \neq 0$

$n = -1$ $\quad n \neq 0$

The number is -1.

[page 393]

18. (a) In one hour press A can do $\frac{1}{3}$ of the job.

In one hour press B can do $\frac{1}{2}$ of the job.

In h hours press A can do $\frac{h}{3}$ of the job.

In h hours press B can do $\frac{h}{2}$ of the job.

$$\frac{h}{3} + \frac{h}{2} = 1$$

(The 1 represents one whole job, which is equal to the sum of the fractions of the job.)

$$\left(\frac{h}{3} + \frac{h}{2}\right)6 = 6$$

$$2h + 3h = 6$$

$$5h = 6$$

$$h = \frac{6}{5}$$

The presses A and B can complete the job together in $1\frac{1}{5}$ hours, or 1 hour and 12 minutes.

(b) If press C takes c hours to do the job alone, then in one hour press C can do $\frac{1}{c}$ of the job; in 2 hours press C can do $\frac{2}{c}$ of the job; in 2 hours press A can do $\frac{2}{3}$ of the job.

$$\frac{2}{3} + \frac{2}{c} = 1 \quad \text{and } c \neq 0$$

$$c = 6$$

It would take press C 6 hours alone.

(c) If press A works a hours after B stops, then

$$\frac{1}{2} + \frac{1}{3} + \frac{a}{3} = 1$$

$$a = \frac{1}{2}$$

Press A takes $\frac{1}{2}$ hour to finish the job.

[pages 393-394]

19. (a) $A = \frac{1}{2}bh$

$$2A = bh$$

$$\frac{2A}{b} = h \quad \text{and} \quad b \neq 0$$

(b) $T = \frac{D}{R} \quad \text{and} \quad R \neq 0$

$$TR = D \quad \text{and} \quad R \neq 0$$

$$R = \frac{D}{T} \quad \text{and} \quad T \neq 0, \; R \neq 0$$

(c) $A = \frac{1}{2}h(x + y)$

$$2A = h(x + y)$$

$$\frac{2A}{x + y} = h \quad \text{and} \quad x + y \neq 0$$

(d) $S = \frac{n}{2}(a + \ell)$

$$2S = na + n\ell$$

$$2S - na = n\ell$$

$$\frac{2S - na}{n} = \ell \quad \text{and} \quad n \neq 0$$

or $\quad S = \frac{n}{2}(a + \ell)$

$$\frac{2S}{n} = a + \ell \quad \text{and} \quad n \neq 0$$

$$\frac{2S}{n} - a = \ell \quad \text{and} \quad n \neq 0$$

(e) $\frac{1}{a} + \frac{1}{b} = 1 \quad \text{and} \quad a \neq 0, \; b \neq 0$

$$\left(\frac{1}{a} + \frac{1}{b}\right)ab = ab \quad \text{and} \quad a \neq 0, \; b \neq 0$$

$$b + a = ab \quad \text{and} \quad a \neq 0, \; b \neq 0$$

$$a = ab - b \quad \text{and} \quad a \neq 0, \; b \neq 0$$

$$a = (a - 1)b \quad \text{and} \quad a \neq 0, \; b \neq 0$$

$$\frac{a}{a - 1} = b \quad \text{and} \quad a \neq 0, \; b \neq 0, \; a \neq 1.$$

[page 394]

13-5. Squaring.

If $a = b$, then a and b are names for the same number. If that number is squared, a^2 and b^2 are names for the new number and $a^2 = b^2$. Some people like the more formal way of saying this; if $a = b$ then $a^2 = ab$ and $ab = b^2$; so $a^2 = b^2$ by the transitive property of equality.

This does not work in reverse because there are two square roots of a^2 and of b^2. Thus we could say that $(-3)^2 = (3)^2$, but $-3 \neq 3$.

Here is a chain of equivalent sentences.

$$a^2 = b^2$$

$a^2 - b^2 = 0$ Addition property of equality

$(a - b)(a + b) = 0$ Factoring

$a - b = 0$ or $a + b = 0$ $xy = 0$ if and only if $x = 0$ or $y = 0$

$a = b$ or $a = -b$ Addition property of equality

It is apparent that squaring both sides of an equation usually does not yield an equivalent equation. And yet in solving certain equations involving square roots or absolute values we need to square both sides. We do so then, bearing carefully in mind that we may expect to find a larger truth set in the new equation. We must therefore test the members of this truth set to find which ones really make the original equation true.

Answers to Problem Set 13-5a; page 395:

1. $x^2 = 4$ has the truth set $\{2,-2\}$, whereas
   $x = 2$ has the truth set $\{2\}$.
2. $(x - 1)^2 = 1^2$ has the truth set $\{0,2\}$, whereas
   $x - 1 = 1$ has the truth set $\{2\}$.
3. $(x + 2)^2 = 0$ has the truth set $\{-2\}$, and
   $x + 2 = 0$ has the same truth set.

[pages 394-395]

4. $(x - 1)^2 = 2^2$ has the truth set $\{3,-1\}$, whereas
$x - 1 = 2$ has the truth set $\{3\}$.

Answers to Problem Set 13-5b; pages 397-398:

1. $\sqrt{2x} = 1 + x$

$2x = 1 + 2x + x^2$

$0 = 1 + x^2$

The truth set is $\emptyset$.

2. $\sqrt{2x + 1} = x + 1$

$2x + 1 = x^2 + 2x + 1$

$0 = x^2$

$x = 0$

If $x = 0$, the left member: $\sqrt{2\cdot 0 + 1} = 1$

the right member: $0 + 1 = 1$

The truth set is $\{0\}$.

3. $\sqrt{x + 1} - 1 = x$

$\sqrt{x + 1} = x + 1$

$x + 1 = x^2 + 2x + 1$

$0 = x^2 + x$

$0 = x(x + 1)$

$x = 0$ or $x = -1$

If $x = 0$, the left member: $\sqrt{0 + 1} - 1 = 0$

the right member: 0

If $x = -1$, the left member: $\sqrt{-1 + 1} -1 = -1$

the right member: -1

The truth set is $\{0,-1\}$.

4. $\sqrt{4x} - x + 3 = 0$

$\sqrt{4x} = x - 3$

$4x = x^2 - 6x + 9$

$0 = x^2 - 10x + 9$

$0 = (x - 9)(x - 1)$

$x - 9 = 0$ or $x - 1 = 0$

$x = 9$ or $x = 1$

If $x = 9$, the left member: $\sqrt{4 \cdot 9} - 9 + 3 = 0$

the right member: 0

If $x = 1$, the left member: $\sqrt{4 \cdot 1} - 1 + 3 = 4$

the right member: 0

The truth set is $\{9\}$.

5. $3\sqrt{x + 13} = x + 9$

$9(x + 13) = x^2 + 18x + 81$

$9x + 117 = x^2 + 18x + 81$

$0 = x^2 + 9x - 36$

$0 = (x + 12)(x - 3)$

$x = -12$ or $x = 3$

If $x = -12$, the left member: $3\sqrt{-12 + 13} = 3$

the right member: $-12 + 9 = -3$

If $x = 3$, the left member: $3\sqrt{3 + 13} = 12$

the right member: $3 + 9 = 12$

The truth set is $\{3\}$.

6. $|2x| = x + 1$

$4x^2 = x^2 + 2x + 1$

$3x^2 - 2x - 1 = 0$

$(3x + 1)(x - 1) = 0$

$3x + 1 = 0$ or $x = 1$

$x = -\frac{1}{3}$ or $x = 1$

If $x = -\frac{1}{3}$, the left member: $|2(-\frac{1}{3})| = \frac{2}{3}$

the right member: $-\frac{1}{3} + 1 = \frac{2}{3}$

If $x = 1$, the left member: $|2\cdot 1| = 2$

the right member: $1 + 1 = 2$

The truth set is $\{-\frac{1}{3}, 1\}$.

7. $2x = |x| + 1$

$2x - 1 = |x|$

$4x^2 - 4x + 1 = x^2$

$3x^2 - 4x + 1 = 0$

$(3x - 1)(x - 1) = 0$

$x = \frac{1}{3}$ or $x = 1$

If $x = \frac{1}{3}$, the left member: $2\cdot\frac{1}{3} = \frac{2}{3}$

the right member: $|\frac{1}{3}| + 1 = \frac{4}{3}$

If $x = 1$, the left member: $2.1 = 2$

the right member: $|1| + 1 = 2$

The truth set is $\{1\}$.

8. $x = |2x| + 1$

$x - 1 = |2x|$

$x^2 - 2x + 1 = 4x^2$

$0 = 3x^2 + 2x - 1$

$0 = (3x - 1)(x + 1)$

$x = \frac{1}{3}$ or $x = -1$

If $x = \frac{1}{3}$, the left member: $\frac{1}{3}$

the right member: $\left|2\cdot\frac{1}{3}\right| + 1 = \frac{5}{3}$

If $x = -1$, the left member: -1

the right member: $|2(-1)| + 1 = 3$

The truth set is $\emptyset$.

9. $x - |x| = 1$

$x - 1 = |x|$

$x^2 - 2x + 1 = x^2$

$-2x + 1 = 0$

$x = \frac{1}{2}$

If $x = \frac{1}{2}$, the left member: $\frac{1}{2} - \left|\frac{1}{2}\right| = 0$

the right member: 1

The truth set is $\emptyset$.

10. $|x - 2| = 3$

$x^2 - 4x + 4 = 9$

$x^2 - 4x - 5 = 0$

$(x - 5)(x + 1) = 0$

$x = 5$ or $x = -1$

If $x = 5$, the left member: $|5 - 2| = 3$

the right member: 3

If $x = -1$, the left member: $|-1 -2| = 3$

the right member: 3

The truth set is $\{5,-1\}$.

11. For every real number x, $|x|^2 = x^2$.

Proof: If $x \geq 0$, $|x| = x$

so $|x|^2 = x^2$

If $x < 0$, $|x| = -x$

so $|x|^2 = (-x)^2$;

but $(-x)^2 = x^2$ $(-a)(-b) = ab$

so $|x|^2 = x^2$

12. $|x - 3| = x + 2$

$$x^2 - 6x + 9 = x^2 + 4x + 4$$

$$5 = 10x$$

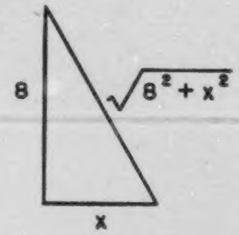
$$\frac{1}{2} = x$$

If $x = \frac{1}{2}$, the left member: $\left|\frac{1}{2} - 3\right| = \frac{5}{2}$

the right member: $\frac{1}{2} + 2 = \frac{5}{2}$

The truth set is $\{\frac{1}{2}\}$.

13. If the other leg is x inches long, the hypotenuse is $\sqrt{8^2 + x^2}$ inches long. (Since $(\text{hypotenuse})^2 = 8^2 + x^2$)



$$\sqrt{8^2 + x^2} = (8+ x) - 4$$

$$\sqrt{64 + x^2} = 4 + x$$

$$64 + x^2 = 16 + 8x + x^2$$

$$48 = 8x$$

$$6 = x$$

If $x = 6$, the left member: $\sqrt{8^2 + 6^2} = \sqrt{64 + 36} = \sqrt{100} = 10$

the right member: $(8 + 6) - 4 = 14 - 4 = 10$

The other leg is 6 inches long.

14. $t = \sqrt{\frac{2s}{g}}$

$$t^2 = \frac{2s}{g}$$

$$t^2g = 2s$$

$$\frac{t^2g}{2} = s$$

If $t = 6.25$ and $g = 32$,

$$s = \frac{(6.25)^2 \cdot 32}{2}$$

$$s = \frac{(\frac{25}{4})^2 \cdot 32}{2}$$

$$s = 625$$

$$\sqrt{\frac{2 \cdot 625}{32}} = \sqrt{\frac{625}{16}}$$

$$= \frac{25}{4}$$

$$= 6.25$$

15. $t = \sqrt{\frac{2s}{g}}$

$$t^2 = \frac{2s}{g}$$

$$t^2g = 2s$$

$$g = \frac{2s}{t^2}$$

This, of course, assumes that $t > 0$, $s \geq 0$, $g > 0$.

If $g = \frac{2s}{t^2}$, $\sqrt{\frac{2s}{g}} = \sqrt{\frac{2s}{\frac{2s}{t^2}}}$

$$= \sqrt{\frac{2st^2}{2s}}$$

$$= \sqrt{t^2}$$

$$= t$$

16. (a) Not equivalent. The numbers $x = 0$, $y = -1$ satisfy the first sentence but not the second.

(b) Equivalent. For every pair of numbers for which the first sentence is true, the second sentence also is true since $\sqrt{1} = 1$. For every pair of numbers for which the second sentence is true, the first sentence is true since, if $a = b$, then $a^2 = b^2$.

(c) Chain of equivalent sentences:

$x^2 = xy$

$x^2 - xy = 0$

$x(x - y) = 0$

$x = 0$ or $x - y = 0$

*13-6. Polynomial Inequalities.

The statement about when a product of several non-zero numbers is positive and when it is negative can, of course, be proved by using the commutative and associative properties to group the negative factors in pairs. The product of each pair of negative factors is positive, and the product of these pairs and all the positive factors will still be positive. Hence, the whole product will be negative only when there is an odd number of negative factors. The class should be encouraged by discussion to fill in these details.

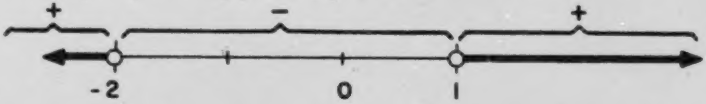
Page 399. For instance, when $x = 2$, it is sufficient to recognize that $(2 + 3)$ is positive, $(2 + 2)$ is positive, $(2 - 1)$ is positive. Hence, the product is positive. Similarly for $x = -\frac{5}{2}$, $(-\frac{5}{2} + 3)$ is positive, $(-\frac{5}{2} + 2)$ is negative, $(-\frac{5}{2} - 1)$ is negative. Since, there are two negative factors, the product is positive.

Page 399. The truth set of $(x + 3)(x + 2)(x - 1) > 0$ is the set of all x such that $-3 < x < -2$ or $x > 1$.

The truth set of $(x + 3)(x + 2)(x - 1) \geq 0$ is the set of all x such that $-3 \leq x \leq -2$ or $x \geq 1$.

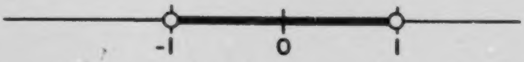
Answers to Problem Set 13-6a; page 401:

1. (a) $(x - 1)(x + 2) > 0$



The set of numbers less than -2 or greater than 1.

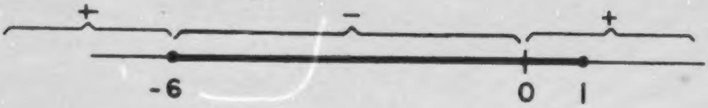
(b) $y^2 < 1$



The set of numbers less than 1 and greater than -1.

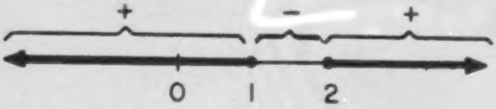
[pages 399-401]

(c) $t^2 + 5t \leq 6$



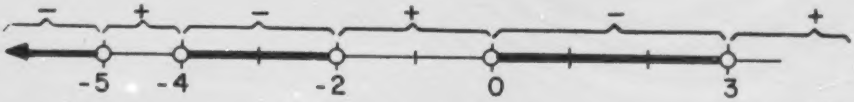
The set of numbers (greater than or equal to -6) and (less than or equal to 1).

(d) $x^2 + 2 \geq 3x$



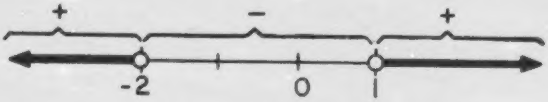
The set of numbers (less than or equal to 1) or (greater than or equal to 2).

(e) $(s + 5)(s + 4)(s + 2)(s)(s - 3) < 0$



The set of numbers (less than -5) or (greater than -4 and less than -2) or (greater than 0 and less than 3).

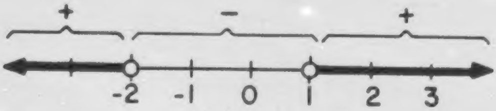
(f) $2 - x^2 < x$



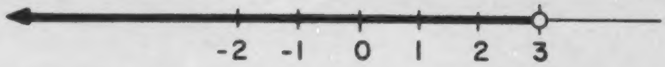
The set of numbers less than -2 or greater than 1.

2. $(x + 1)(x - 1) > 0$ and $x < 3$

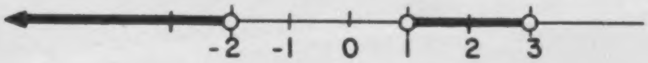
The graph of $(x + 2)(x - 1) > 0$ is



The graph of $x < 3$ is



The truth set of "$(x + 2)(x - 1) > 0$ and $x > 3$" is the set of all numbers each of which is in both the above truth sets.



The set of numbers (less than -2) or (greater than 1 and less than 3).

*3. If $(x + 2)(x - 1)$ is positive and $x - 3$ is negative then $(x + 2)(x - 1)(x -3)$ is negative; thus, every solution of the sentence in Problem 2 is a solution of $(x + 2)(x - 1)(x - 3) < 0$. This inequality is therefore a likely candidate, and when we graph its truth set we get the graph drawn in Problem 2, so that the two sentences are equivalent.

Page 401. If x is a solution of $(x + 2)^2(x - 1) > 0$ then $(x + 2)^2$ must not be 0 and therefore must be positive, being a square. Multiplying by the positive number $\frac{1}{(x + 2)^2}$ we obtain $x - 1 > 0$. Going backwards we see that, if $x - 1 > 0$, $x > 1$ and hence $x \neq -2$, so that $x + 2 \neq 0$ and so $(x + 2)^2$ must be positive. Multiplying $x - 1 > 0$ by this positive number gives us $(x + 2)^2(x - 1) > 0$. Hence $(x + 2)^2(x - 1) > 0$ and $x - 1 > 0$ are equivalent sentences.

The truth set of $(x + 2)^2(x - 1) \leq 0$ is the set of all numbers that are not in the truth set of $(x + 2)^2(x - 1) > 0$. The latter set we've just seen to be all x such that $x > 1$. Thus the truth set of $(x + 2)^2(x - 1) \leq 0$ is the set of all x such that $x \leq 1$.

The product of x and $(x - 1)^3$ will be negative if and only if either $x < 0$ and $(x - 1)^3 > 0$ or $x > 0$ and $(x - 1)^3 < 0$.

The first clause is equivalent to "$x < 0$ and $x - 1 > 0$", that is, to "$x < 0$ and $x > 1$". Since no number is both less than 0 and greater than 1, this sentence has no solution. The second clause is equivalent to "$x > 0$ and $x - 1 < 0$" and this has its truth set consisting of all x between 0 and 1. The truth set of $x(x - 1)^3 < 0$ is thus the set $0 < x < 1$.

The truth set of $x(x - 1)^3 \geq 0$ will be all the numbers not in the truth set of $x(x - 1)^3 < 0$. The latter set we have just seen to be the set $0 < x < 1$. The numbers not in this set consist of all $x \leq 0$ together with all $x \geq 1$. Therefore, "$x(x - 1)^3 \geq 0$" is equivalent to "$x \leq 0$ or $x \geq 1$".

We have used above the fact that where the factor $x - 1$ occurs three times there are three factors changing together from negative to positive as x crosses 1, so their product changes from negative to positive. Some students may enjoy extending this idea to polynomials with the same factor four or five times, and then generalizing the situation.

A factor which is a positive real number such as $x^2 + 2$, will not change the product from a negative to a positive number or vice versa. For this reason the truth set of $(x^2 + 2)(x - 3) < 0$ is the truth set of $x - 3 < 0$, that is, all x such that $x < 3$; and the truth set of $(x^2 + 2)(x - 3) \geq 0$ is the set of all x such that $x \geq 3$.

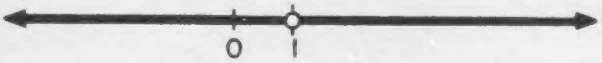
Answers to Problem Set 13-6b; page 402 :

1. $x^2 + 1 > 2x$

$x^2 + 1 - 2x > 0$

$(x - 1)^2 > 0$

Since $a^2 > 0$ for all real numbers a except zero, the truth set is the set of all real numbers except 1.

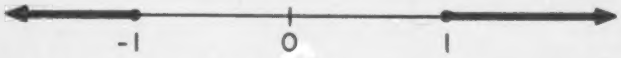


2. $x^2 + 1 < 0$

The truth set is $\emptyset$.

3. $(t^2 + 1)(t^2 - 1) \geq 0$

The set of numbers (less than or equal to -1) or (greater than or equal to 1).

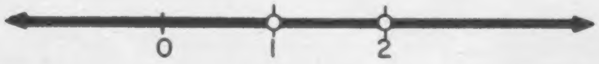


4. $4s - s^2 > 4$

The truth set is $\emptyset$.

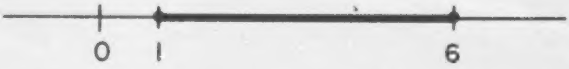
5. $(x - 1)^2(x - 2)^2 > 0$

The set of all real numbers except 1 and 2.



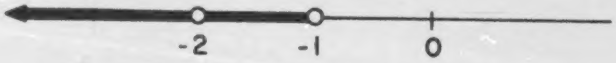
6. $(y^2 - 7y + 6) \leq 0$

The set of numbers (greater than or equal to 1) and (less than or equal to 6).



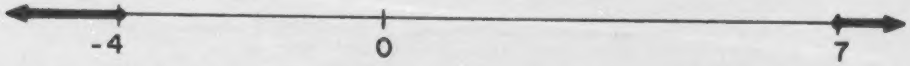
7. $(x + 2)(x^2 + 3x + 2) < 0$

The set of numbers less than -1, except -2.



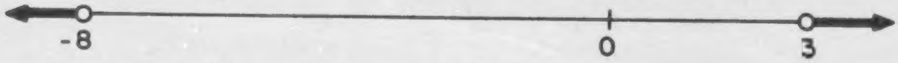
8. $3y + 12 \leq y^2 - 16$

The set of numbers (less than or equal to -4) or (greater than or equal to 7).



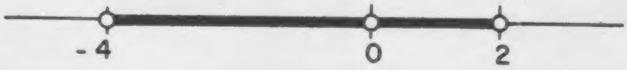
9. $x^2 + 5x > 24$

The set of numbers less than -8 or greater than 3.



10. $|x| \ (x - 2)(x + 4) < 0$

The set of numbers greater than -4 and less than 2, except 0.



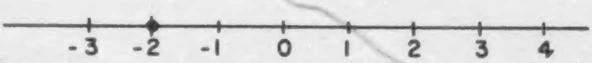
Answers to Review Problems

1. yes, $\frac{1}{x^2 + 1}$ is a real number for every x.
2. no, $\frac{x^2 - 1}{x^2 - 1} \neq 1$ for $x = 1$ or $x = -1$.
3. yes, $\frac{1}{x^2 + 4}$ is a real number for every x.
4. no, $\frac{x}{x - 3} - \frac{3}{x - 3} = 0$ is not a sentence for $x = 3$.
5. no, $|x| = 2$ has the truth set $\{2, -2\}$ but 2 is not a solution of $\frac{x + 2}{x - 2} = 0$.
6. yes
7. $\emptyset$
8. $\{-3\}$
9. $\{3,6\}$
10. $\{1,-1\}$
11. $\{0,1,2\}$
12. $\{-4\}$
13. $\emptyset$
14. $\emptyset$, notice that $\sqrt{x + 2} = -2$ cannot be true for any real number since it asserts that a positive number is the same as a negative number.
15. $\{2\}$
16. $\{2,-4\}$
17. $\{\frac{1}{2}\}$
18. $\emptyset$, notice that if $x \geq 0$, $1 = 0$ and if $x < 0$, $x = \frac{1}{2}$ are both contradictions.
19. Every x except $x = -1$.
20. Every x.

[pages 403-404]

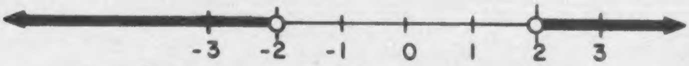
21. (a) The truth set is $\{-2\}$.

The graph:



(b) The truth set is each x such that $x < -2$ or $x > 2$.

The graph:



(c) Same as (b)

(d) Same as (b)

22. $\sqrt{1 + 2x} < x - 1$

We observe that $\sqrt{1 + 2x}$ is defined for $x \geq -\frac{1}{2}$ and that if there is an x such that $x - 1$ is greater than a non-negative number, then $x > 1$.

Thus:

$$\sqrt{1 + 2x} < x - 1 \quad \text{and} \quad x > 1$$

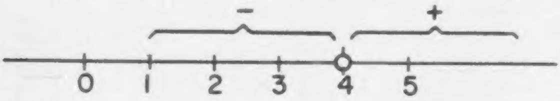
is equivalent to

$$1 + 2x < x^2 - 2x + 1 \quad \text{and } x > 1$$

is equivalent to

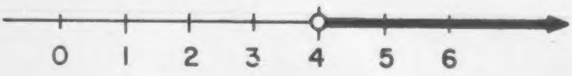
$$0 < x(x - 4) \quad \text{and} \quad x > 1.$$

We need consider only values of $x > 1$.



Thus, the truth set consists of every number greater than 4.

The graph:

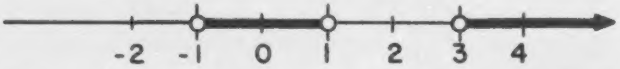


The first two sentences are reversible since

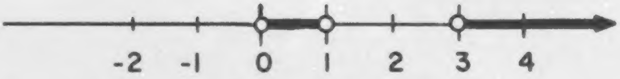
(i) If $a < b$, and a and b are positive then $a^2 < b^2$

(ii) If $0 < a < b$, then $\sqrt{a} < \sqrt{b}$.

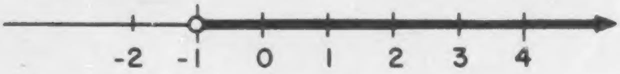
23. (a)



(b)



(c)



Suggested Test Items

1. For each pair of sentences, determine whether the two sentences are equivalent

   (a) $3x + 6 = 8, \quad x = \frac{2}{3}$

   (b) $\frac{x - 2}{x + 2} = 0, \quad x = 2$

   (c) $|x| + 1 = 4, \quad x^2 - 9 = 0$

2. For each pair of sentences, determine whether the two sentences are equivalent.

   (a) $x(x - 3) + 3(x - 3) = 0, \quad x - 3 = 0$

   (b) $x(x - 3) - 3(x - 3) = 0, \quad x - 3 = 0$

3. Find the truth set of each of the following sentences.

   (a) $x(x - 3) + 3(x - 3) = 0$

   (b) $x(x - 3) - 3(x - 3) = 0$

4. Find the truth set of $\sqrt{1 - 2x} = x - 1$.

5. Solve: $\sqrt{x^2 - 9} = 4$

6. Graph the truth set of each of the following sentences. Describe the truth set.

   (a) $8y - 3 > 3y + 7$

   (b) $|x| < 1 - x$

7. Graph the truth set of each of the following compound sentences.

   (a) $x - 3 < 0$ and $x \geq 0$

   (b) $x - 3 < 0$ or $x \geq 0$

   (c) $x - 3 > 0$ and $x \leq 0$

   (d) $x - 3 > 0$ or $x \leq 0$

8. Describe and graph the truth set of

$$\frac{m}{m - 2} \leq 3$$

9. Solve and graph

$$\frac{3}{y^2 - 1} < 1$$

10. Solve and graph

$$\frac{1}{x^2 + 1} < 1$$

11. Describe and graph the truth set of

$$x^2 + 1 < 2x + 1$$

Chapter 14

GRAPHS OF OPEN SENTENCES IN TWO VARIABLES

In this chapter we extend graph work from the line to the plane by introducing coordinate axes and associating points of the plane with ordered pairs of numbers. We draw the graphs of the truth sets of sentences in two variables, both equations and inequalities, with especial attention at first to linear expressions. We include graphs of open sentences which involve absolute value. For the better students we give some attention to reflection of the points of the plane about an axis, and movement of points in the plane, and the effect of these changes on the equation of the graph.

Students who have studied the S.M.S.G. 8th Grade Course will have some familiarity with the rectangular coordinate system in the plane and some simple graphs. Most of this chapter, however, will be new to them.

The teacher is referred to Studies in Mathematics, Volume III, pages 6.8-6.17, for a discussion of open sentences in two variables.

14-1. The Real Number Plane.

Page 405. We hope to put enough emphasis on the ordering of the pairs of numbers, both now and later, so that this is perfectly natural to the student. That is why we start with one number line.
Page 406. Here again we work on the ordering. The pupil should be expected to state, for points P, A, B, L, and Q, that the number written first is the one associated with the horizontal number line, and the one written second is the one associated with the vertical line.

The number pair for Q differs from that for P in that the second number for Q is negative, while the second number for P is positive. The second number for Q is negative because it is measured down from the horizontal number line, while the second number for P is measured up.

The ordered pair associated with E is (5,4), with C is (-2,-5), with K is (0,-5), with D is (3,-6). The ordered pair associated with H is (0,0), with F is (8,0), and with G is (-4,0). If a point lies on the horizontal line, the second number of the ordered pair associated with the point is 0.

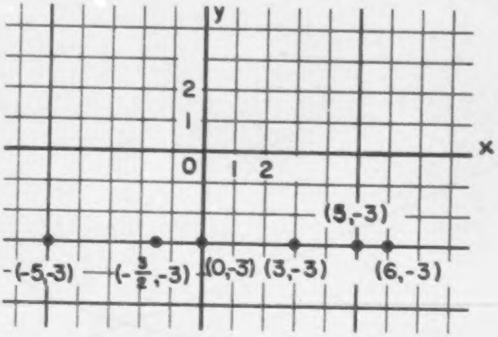
Page 407. Here again, in pointing out the difference in the ordered pairs of numbers associated with S and T we emphasize the order.

Answers to Problem Set 14-1a; page 408:

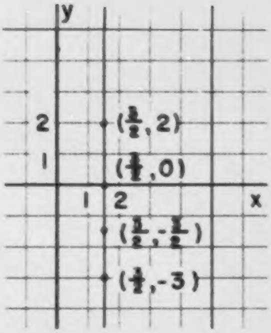
1. A(6,-6)
B(-5,-5)
C(3,-4)
D(-8,-4)
E(2,3)
F(4,6)
G(-3,0)
H(0,-6)
$I(-4,5\frac{1}{2})$
$J(7\frac{1}{2},3)$
K(-8,4)
L(7,0)
M(0,7)

2. Point out to the pupils the use of Roman numerals in the numbering of the quadrants. The points for which the second coordinate is equal to the first lie in quadrants I and III.

3. All the points whose ordinates are -3 lie 3 units below the x-axis. They form a straight line.



4. All the points whose abscissas are $\frac{3}{2}$ lie on the line $1\frac{1}{2}$ units to the right of the y-axis.


y
2
1
1
2
x
$(\frac{3}{2}, 2)$
$(\frac{3}{2}, 0)$
$(\frac{3}{2}, -\frac{1}{2})$
$(\frac{3}{2}, -3)$

Answers to Problem Set 14-1b; pages 409-411:

1.

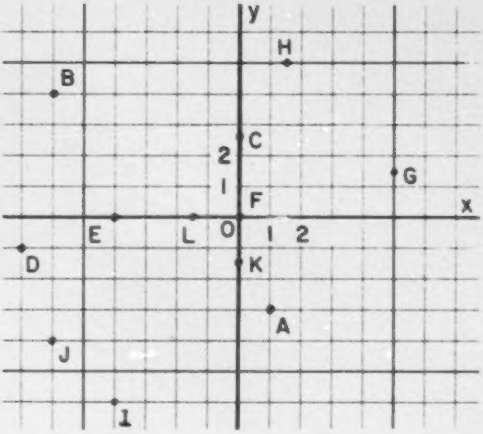

y
H
B
C
2
G
1
F
x
E
L
O
1
2
D
K
A
J
I

Figure for Problem 1.

2. G and H are not the same point, because although the same numbers are used in the coordinates, the order is different. For the same reason, I and J are not the same point, nor are K and L.

3. All of the ordered pairs of numbers have the abscissa 2. All of the points for which the abscissa of the ordered pair is 2 lie on the line parallel to the y-axis and 2 units to the right of it.

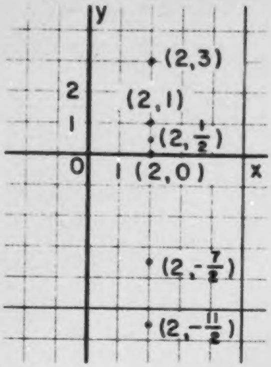


Figure for Problem 3.

4. All of the ordered pairs having 5 for their ordinates are associated with points which lie on the line parallel to the y-axis and 5 units above it.

5. If you could locate all of the points whose coordinates are pairs of numbers for which the first and second number are the same, you would have a straight line through the origin.

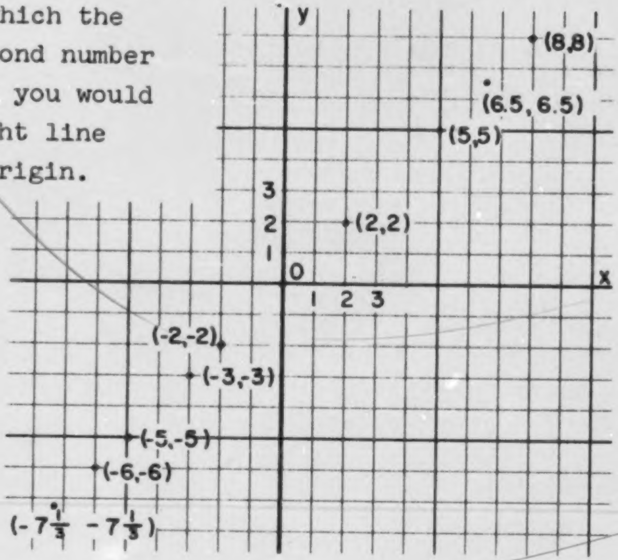


Figure for Problem 5.

[pages 409-410]

*6. Each point is moved to a point with the same ordinate, but whose abscissa is the opposite of the abscissa of the original point.

(a) (2,1) goes to (-2,1)
(2,-1) goes to (-2,-1)
$(-\frac{1}{2},2)$ goes to $(\frac{1}{2},2)$
(-1,-1) goes to (1,-1)
(3,0) goes to (-3,0)
(-5,0) goes to (5,0)
(0,2) goes to (0,2)
(0,-2) goes to (0,-2)

(b) (-2,1) goes to (2,1)
(-2,-1) goes to (2,-1)
$(\frac{1}{2},2)$ goes to $(-\frac{1}{2},2)$
(1,-1) goes to (-1,-1)
(-3,0) goes to (3,0)
(5,0) goes to (-5,0)
(0,2) goes to (0,2)
(0,-2) goes to (0,-2)

(c) (c,-d) goes to (-c,-d)

(d) (-c,d) goes to (c,d)

(e) (c,d) goes to (-c,d)

(f) The points on the y-axis go to themselves.

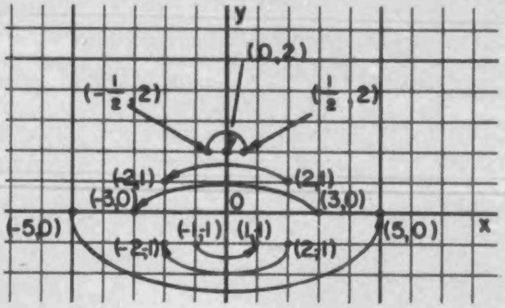

y
(0,2)
$(-\frac{1}{2},2)$
$(\frac{1}{2},2)$
(-2,1)
(2,1)
(-3,0)
O
(3,0)
(-5,0)
(-1,-1)
(1,-1)
(5,0)
x
(-2,-1)
(2,-1)

Figure for Problem 6.

*7.

(a) (1,1) goes to (3,1)
(-1,1) goes to (1,1)
(-2,2) goes to (0,2)
(0,-3) goes to (2,-3)
(3,0) goes to (5,0)

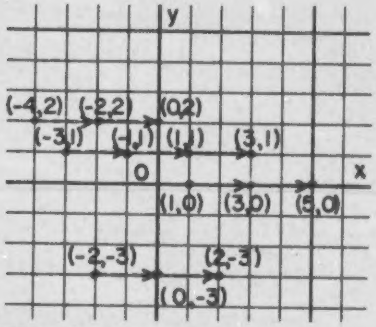

y
(-4,2)
(-2,2)
(0,2)
(-3,1)
(-1,1)
(1,1)
(3,1)
O
x
(1,0)
(3,0)
(5,0)
(-2,-3)
(2,-3)
(0,-3)

Figure for Problem 7.

(b) $(-1,1)$ goes to $(1,1)$
$(-3,1)$ goes to $(-1,1)$
$(-4,2)$ goes to $(-2,2)$
$(-2,-3)$ goes to $(0,-3)$
$(1,0)$ goes to $(3,0)$

(c) $(c-2,d)$ goes to (c,d)

(d) $(-c-2,d)$ goes to $(-c,d)$

(e) No points go to themselves.

14-2. Graphs of Open Sentences With Two Variables.

Here our object is to establish the connection between ordered pairs as associated with points in the plane, and ordered pairs as solutions of open sentences. Again the emphasis is on the order.

Page 411. If 0 is assigned to y and -2 to x, we have

$3(0) - 2(-2) + 6 = 0$. This sentence is not true.

If 0 is assigned to x and -2 to y, we have

$3(-2) - 2(0) + 6 = 0$. This sentence is true.

Page 411. As seen above, $(0,-2)$ belongs to the truth set of the sentence

$$3y - 2x + 6 = 0$$

while $(-2,0)$ does not belong to the truth set.

Page 412. If r is taken as the first variable, solutions of "$s = r + 1$" include $(0,1)$, $(-5,-4)$, $(2\frac{1}{3},3\frac{1}{3})$ and so on. $(-2,-3)$ is not a solution but $(-3,-2)$ is a solution.

If u is taken as the first variable, solutions of "$v = 2u^2$" include $(0,0)$, $(2,8)$, $(-1,2)$, $(.5,.5)$ and so on. $(-1,2)$ is a solution; $(2,-1)$ does not satisfy the sentence.

Solutions of "$y = 4$" as a sentence in two variables include: $(0,4)$, $(-3,4)$, $(5.3,4)$ and so on. Every ordered pair satisfying the sentence has 4 as its ordinate.

Every ordered pair satisfying the sentence "$x = -2$" has -2 as its abscissa.

Answers to Problem Set 14-2a; page 413:

1. (a) The truth set is the set of all ordered pairs whose ordinates are 5.

   (b) The truth set is the set of all ordered pairs whose abscissas are 0.

   (c) The truth set is the set of all ordered pairs such that the ordinate is -3 times the abscissa.

   (d) The truth set is the set of all ordered pairs whose abscissas are 3.

2. Suggested pairs are given here. Have each pupil check his choices of solution by verifying to see that they satisfy the open sentences:

(a)	$(0,-2)$,	$(2,4)$,	$(-2,-8)$,	$(\frac{3}{2},\frac{5}{2})$
(b)	$(-2,0)$,	$(0,2)$,	$(\frac{7}{2},\frac{11}{2})$,	$(-5,-3)$
(c)	$(-3,10)$,	$(0,1)$,	$(3,10)$,	$(\frac{4}{3},\frac{25}{9})$
(d)	$(-1,1)$,	$(0,0)$,	$(2,2)$,	$(\frac{4}{5},\frac{4}{5})$

3.

(a)	$(1,2)$,	$(-2,-5)$
(b)	$(0,3)$,	$(-5,4)$
(c)	$(-2,-3)$,	$(3,8)$
(d)	$(-5,-5)$,	$(4,3)$

4. (a)

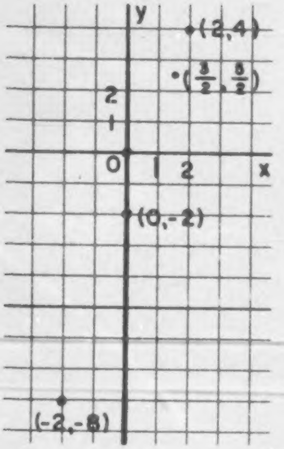


Figure for Problem 4(a).

(b)

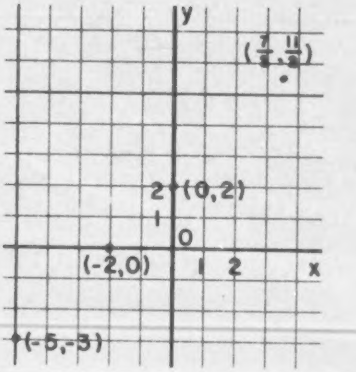


Figure for Problem 4(b).

(c)

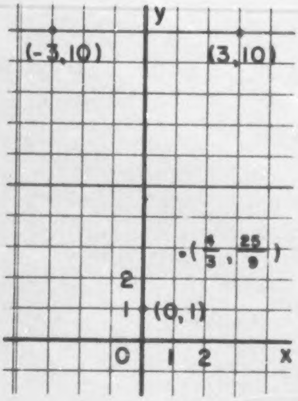


Figure for Problem 4(c).

(d)

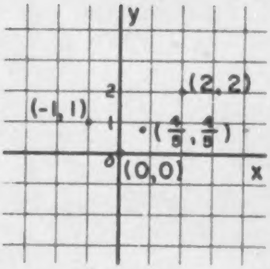


Figure for Problem 4(d).

In doing these, and in the class discussion of them, the pupils will soon note that the points in (a) seem to lie on a straight line as do those in (b), while neither those in (c) nor those in (d) lie on a single straight line. Encourage the pupils to question this - they will find answers further on in the chapter.

Page 414.

x	-9	-6	-3	0	3	5	$16\frac{1}{2}$
y	-8	-6	-4	-2	0	$1\frac{1}{3}$	9

Examination of the graph will show the student that all of the points associated with the ordered pairs indicated in the table do seem to lie on the line.

The coordinates of point A do satisfy the equation, since

$$2(6) - 3(2) - 6 = 0$$

is a true sentence

The general form of the linear equation in two variables

$$Ax + By + C = 0$$

should be stressed, and referred to often, so that the pupils will instantly recognize such equations and will automatically associate them with straight lines. Since the students have not studied geometry formally, we would not expect them to understand a geometric definition of a line. However, their experience with drawing graphs of equations of the form $Ax + By + C = 0$ suggests that we take as our defintion:

> A line is a set of points whose coordinates satisfy an equation of the form
>
> $$Ax + By + C = 0,$$
>
> with not both A and B zero.

In general, every graph (set of points in the plane), even the empty graph, is associated with an open sentence, and conversely.

Notice that we have used "straight line" and "line" interchangeably. Hereafter, we shall prefer the less redundant "line". Be sure that the students understand that lines are "straight" by our mathematical definition, and that "curved line" is a contradiction in terms.

Answers to Problem Set 14-2b; pages 416-417.

The teacher should be sure to insist in all of the exercises which follow that lines be drawn as long as possible within the area of the graph. From the outset the tendency to draw only segments should be discouraged, unless limitations are included in the open sentences.

1. All the points whose ordinates are -3 are on a line parallel to the horizontal axis and 3 units below it.

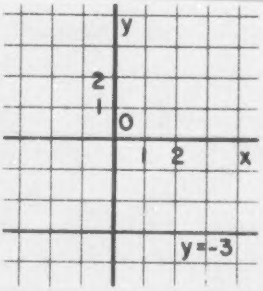


Figure for Problem 1.

2. The equation whose graph is the horizontal axis is "y = 0". The equation whose graph is the vertical axis is "x = 0".

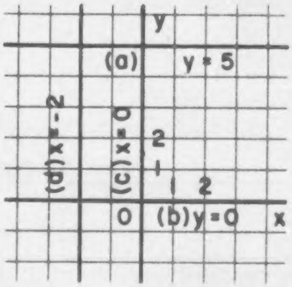


Figure for Problem 2.

3. Line (a) includes all possible points such that each has its abscissa equal to the opposite of the ordinate.
Line (b) includes those points such that each has ordinate twice the abscissa.
Line (c) includes the points such that each has ordinate that is the opposite of twice the abscissa.
All of these graphs are lines, and all pass through the origin. Their equations are:

(a) $y = -x$
(b) $y = 2x$
(c) $y = -2x$

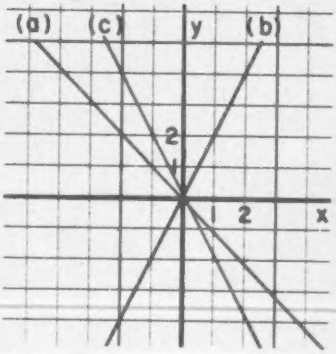


Figure for Problem 3.

4. All of the graphs are lines through the origin. The graph of (a) rises as it goes from left to right, while the graph of (d) descends. The same pattern applies to the graphs of (b) and (e), and also to the graphs of (c) and (f).

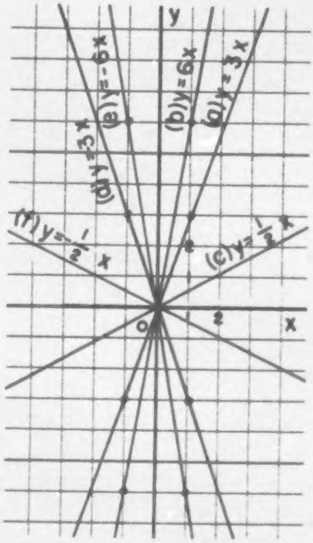


Figure for Problem 4.

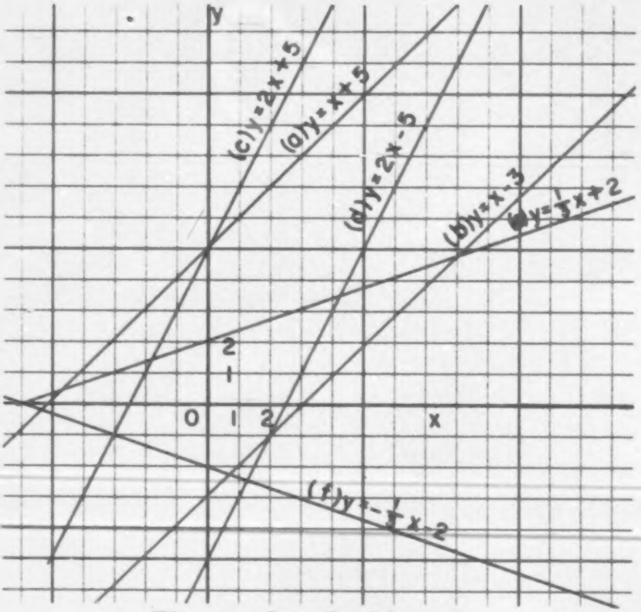


Figure for Problem 5.

5. The graph of (a) differs from the graph of (b) in the fact that it cuts the y-axis at a point 8 units above the point where the graph of (b) cuts it. The graph of (c) cuts the y-axis at a point 10 units above the point where the graph of (d) cuts it. The graph of (e) not only cuts the y-axis at a different point than the point where the graph of (f) cuts it, but also the graph of (e) rises while the graph of (f) descends.

The graphs of (a) and (b) appear to be a pair of parallel lines. The graphs of (c) and (d) also appear to be parallel, but the graphs of (e) and (f) are not.

Page 418. Upon attempting to locate points such as (-2,5), (-1,2), (0,2), (1,4), (2,8), and (3,10), the pupils will soon note that the points are not all on one line, but that they all lie above the line which is the graph of the open sentence "$y = 3x$", since along the y-axis "greater than" means "above". The open sentence whose graph is the set of points for which the ordinate is greater than 3 times the abscissa is "$y > 3x$".

Answers to Problem Set 14-2c; pages 420-422:

1. The open sentence whose truth set is the set of ordered pairs for which the ordinate is two greater than the abscissa is "y= x + 2". The graph of the set of points associated with this set of ordered pairs is the line shown in the figure.

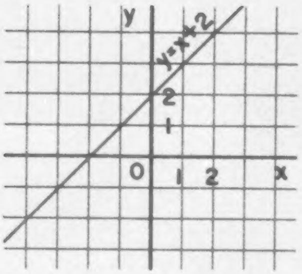


Figure for Problem 1.

It is not possible to draw the graphs of both of the sentences "$y > x + 2$" and "$y \geq x + 2$", because in the first one the line whose equation is "$y = x + 2$" is dotted, and in the second one it is a solid line.

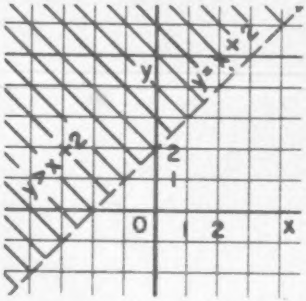


Figure for Problem 1(a).

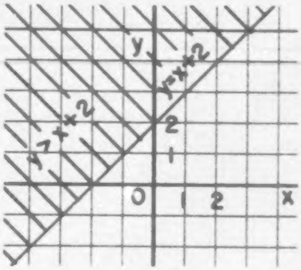


Figure for Problem 1(b).

2. In the sentence "$y = |x|$", since x is positive for all values of x, it follows that y is never negative. The solutions for which the abscissas are given are: $(-3,3), (-1,1), (1\frac{1}{2}, 1\frac{1}{2}), (2,2), (4,4)$.

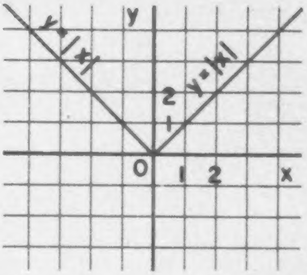


Figure for Problem 2.

3. (a) $y = 2x$

x	-3	-1	0	2	5
y	-6	-2	0	4	10

(b) $y = 3x$

x	-2	-1	0	2	3
y	-6	-3	0	6	9

(c) $y = \frac{1}{2}x$

x	-4	-1	0	2	5
y	-2	$-\frac{1}{2}$	0	1	$2\frac{1}{2}$

(d) $y = -\frac{1}{3}x$

x	-9	-6	0	3	6
y	3	2	0	-1	-2

(e) $y = x$

x	-8	-4	0	3	7
y	-8	-4	0	3	7

(f) $y = -x$

x	-7	-5	0	3	4
y	7	5	0	-3	-4

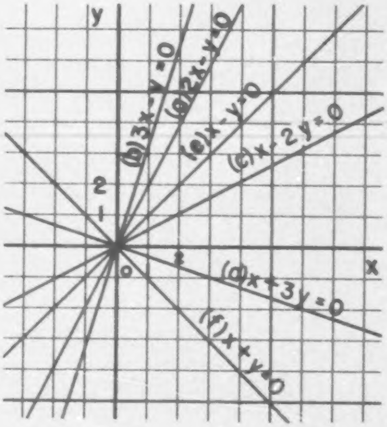


Figure for Problem 3.

The graphs of all of these open sentences are lines through the origin.

[page 420]

4. (a) $y = \frac{3}{2}x$

x	-6	-4	0	4	8
y	-9	-6	0	6	12

(b) $y = \frac{3}{2}x - 3$

x	-4	-2	0	2	4
y	-9	-6	-3	0	3

(c) $y = \frac{3}{2}x - 6$

x	-2	-1	0	4	10
y	-9	$-7\frac{1}{2}$	-6	0	9

(d) $y = \frac{3}{2}x + 3$

x	-6	-2	0	4	6
y	-6	0	3	9	12

(e) $y = \frac{3}{2}x + 6$

x	-8	-6	0	1	2
y	-6	-3	6	$7\frac{1}{2}$	9

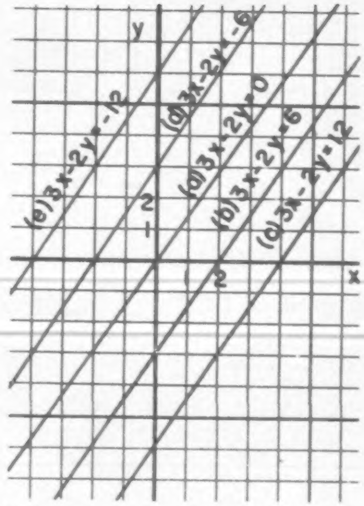


Figure for Problem 4.

The graphs of all of these are lines parallel to each other.

5.

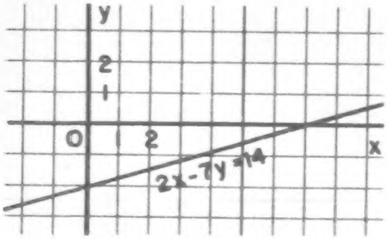


Figure for Problem 5(a).

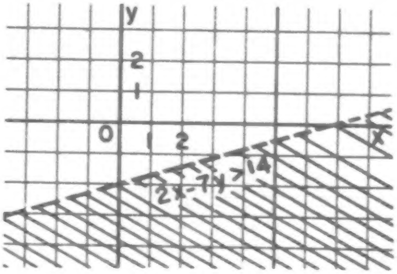


Figure for Problem 5(b).

(a) $2x - 7y = 14$

$y = \frac{2}{7}x - 2$

(b) $2x - 7y > 14$

$y < \frac{2}{7}x - 2$

[page 421]

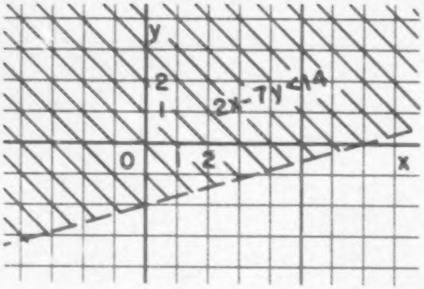


Figure for Problem 5(c).

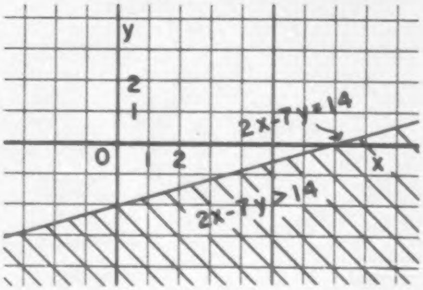


Figure for Problem 5(d).

(c) $2x - 7y < 14$ (d) $2x - 7y \geq 14$

$y > \frac{2}{7}x - 2$ $y \leq \frac{2}{7}x - 2$

In parts (b), (c), and (d), to get the y-form we review the work done in 13-2 on finding the truth sets of inequalities. Have the pupil recall the steps involved:

$2x - 7y > 14$

$-7y > 14 - 2x$ (addition property of order)

$-7y > -2x + 14$ (commutative property of addition)

$y < \frac{2}{7}x - 2$ (multiplication property of order)

6. (a) $5x - 2y = 10$

$$y = \frac{5}{2}x - 5$$

(b) $2x + 5y = 10$

$$y = -\frac{2}{5}x + 2$$

(c) $5x + y = 10$

$$y = -5x + 10$$

(c) $3x - 4y = 6$

$$y = \frac{3}{4}x - \frac{3}{2}$$

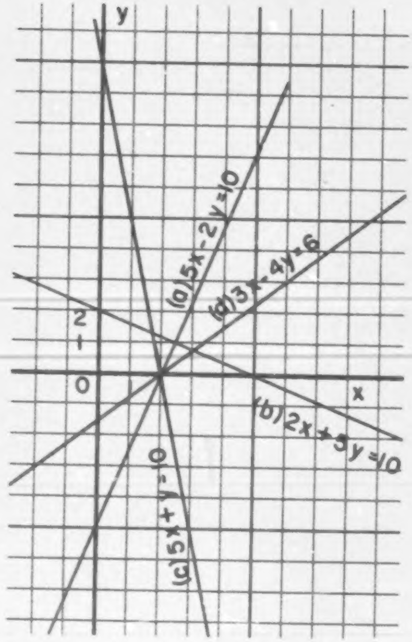


Figure for Problem 6.

In finding points for (d), some class discussion on convenient replacements for x would be in order. The point $(2,0)$ seems to lie on the graphs of (a), (c), and (d). We verify that these coordinates satisfy the open sentences as follows:

(a) $5(2) - 2(0) = 10$ is a true sentence

(c) $5(2) + 0 = 10$ is a true sentence

(d) $3(2) - 4(0) = 6$ is a true sentence.

7. (a) $2x - 3y = 10$

$$y = \frac{2}{3}x - \frac{10}{3}$$

Here it would be a help to the pupil to point out that if he can find one integral replacement for x which produces an integral value for y, then he can find as many others as he wishes by adding to the first value of x multiples of the denominator of the fraction which is the coefficient of x. For example:

x	-4	-1	2	8
y	-6	-4	-2	2

(b) $-x + 2y = \frac{1}{2}$

$$y = \frac{1}{2}x + \frac{1}{4}$$

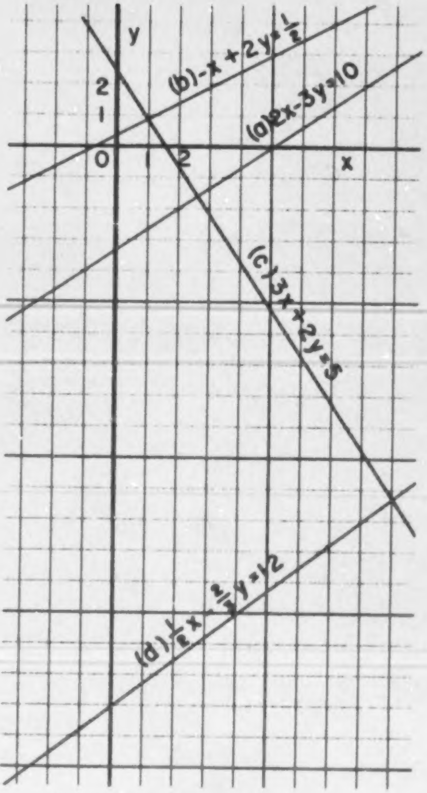


Figure for Problem 7.

Here it is apparent upon inspection that there are no integral values x which produce integral values for y. So we make the best of it:

x	-4	0	3	6
y	$-1\frac{3}{4}$	$\frac{1}{4}$	$1\frac{3}{4}$	$3\frac{1}{4}$

(c) $3x + 2y = 5$

$$y = -\frac{3}{2}x + \frac{5}{2}$$

x	-3	1	5	7
y	7	1	-5	-8

(d) $\frac{1}{2}x - \frac{2}{3}y = 12$

$$y = \frac{3}{4}x - 18$$

x	4	12	16	0
y	-15	-9	-6	-18

8. (a)

x	-3	-2	$-\sqrt{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\sqrt{2}$	2	3
y	9	4	2	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	2	4	9

(b)

x	-3	-2	$-\sqrt{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\sqrt{2}$	2	3
y	-9	-4	-2	-1	$-\frac{1}{4}$	0	$-\frac{1}{4}$	-1	-2	-4	-9

(c)

x	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	$2\frac{1}{2}$	3
y	10	5	2	$1\frac{1}{4}$	1	$1\frac{1}{4}$	2	5	$7\frac{1}{4}$	10

(d)

x	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
y	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-2	-4	no value	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

Note that we approximate $\sqrt{2}$ by 1.4 when drawing graphs.

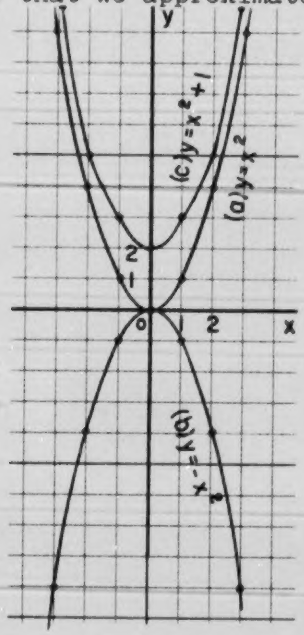


Figure for Problem 8, (a) - (c).

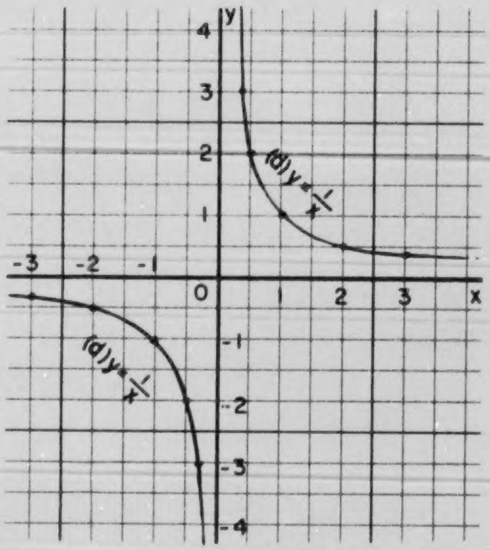


Figure for Problem 8(d).

[page 421]

The graphs of these sentences are not lines, but curves. The open sentences for the first three differ from those considered in previous problems in this chapter in the fact that in each of them the x is squared. In the open sentence for (d), y equals, not a multiple of x, but its reciprocal. So we cannot say that the graph of every open sentence is a line. There is no harm in telling the pupils that the first three graphs are called parabolas, while the fourth is a hyperbola. These will be met again later on.

*9. The object of this problem is to make the pupil aware that a given point may be associated with many different ordered pairs, depending upon the location of the axes.

(a)

Point	(x,y)	(a,b)
P	(-4,2)	(-8,-1)
Q	(-5,6)	(-9,3)
R	(11,8)	(7,5)
S	(-8,-2)	(-12,-5)
T	(8,1)	(4,-2)
U	(2,2)	(-2,-1)
V	(3,-4)	(-1,-7)
W	(12,-2)	(8,-5)

The bright student may quickly discover that for each point "a = x - 4" and "b = y - 3". He should be encouraged to use these facts as a check on his results. If in doing part (b) he uses these facts, then he should check each ordered pair by locating the point on the figure for this problem.

(b)

(x,y)	(a,b)
(5,-5)	(1,-8)
(-3,-4)	(-7,-7)
(-1,0)	(-5,-3)
(3,5)	(-1,2)

14-3. Slopes and Intercepts.

The students need to draw careful graphs of (a) through (j) and find the equation of each of these graphs in preparation for Problem Set 14-3a.

(a)

x	-6	-3	$-2\frac{1}{2}$	0	3	5.1	6
y	-6	-3	$-2\frac{1}{2}$	0	3	5.1	6

When the successive points are connected, they lie on one line. There are no points in the table which do not lie on the line through (-6,-6) and (6,6). The point (8,8) is on the line, but not on the part of it between (-6,-6) and (6,6). The open sentence which describes this graph for all points in the plane is "$y = x$". The line divides the angles formed by the axes into two equal parts.

(b)

x	-6	-5.1	-4.3	0	2.5	4	6.1
y	6	5.1	4.3	0	-2.5	-4	-6.1

It would, of course be easy to determine pairs which fulfill the condition without making the table. One line passes through all of the points. The open sentence which describes this line is "$y = -x$". It differs from the open sentence in (a) because here we associate y with the opposite of x. The pupils might note that this line divides into two equal parts the other pair of angles formed by the axes.

[pages 422,423]

(c) $y = 2x$

(d) $y = 6x$

(e) $y = 3x$

(f) $y = -6x$

(g) $y = -3x$

(h) $y = \frac{1}{2}x$

(i) $y = -\frac{1}{2}x$

(j) $y = \frac{1}{6}x$

(k) $y = -\frac{1}{5}x$

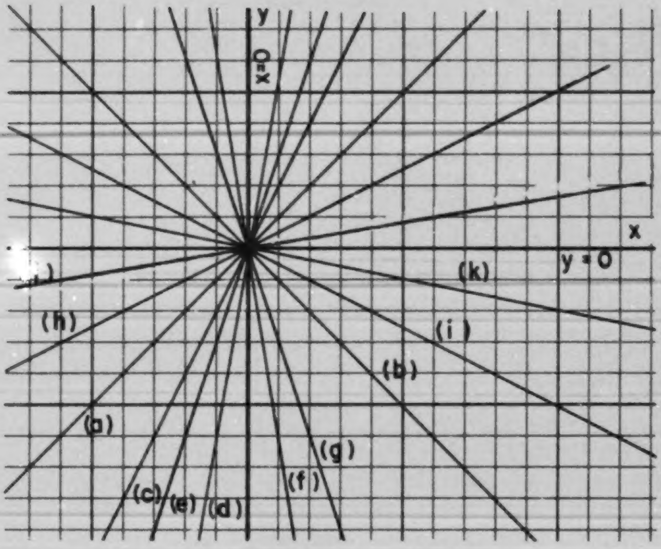


Figure 1.

Answers to Problem Set 14-3a; page 425:

1. The coefficients of x in the open sentences for which the lines lie between the graphs of "$y = x$" and "$x = 0$" are 2, 6, and 3. We observe that all of these numbers are greater than 1.

2. The coefficients of x in the open sentences for which the lines lie between the graphs of "$y = 0$" and "$y = x$" are $\frac{1}{2}$ and $\frac{1}{6}$. These coefficients are greater than 0 but less than 1.

3. The coefficients of x in the open sentences for which the lines lie between the graphs of "$y = 0$" and "$y = -x$" are $-\frac{1}{2}$ and $-\frac{1}{6}$. These coefficients are less than 0 but greater than -1.

4. The coefficients of x in the open sentences for which the lines lie between the graphs of "$y = -x$" and "$x = 0$" are -6 and -3. These coefficients are less than -1.

5. The graph of "$y = .01x$" lies between the graphs of "$y = 0$" and "$y = x$".
The graph of "$y = -100x$" lies between the graphs of "$y = -x$" and "$x = 0$".
The graph of "$y = -56x$" lies between the graphs of "$y = -x$" and "$x = 0$".
The graph of "$y = -\frac{5}{6}x$" lies between the graphs of "$y = -x$" and "$y = 0$".
The graph of "$y = \frac{5x}{12}$" lies between the graphs of "$y = x$" and "$y = 0$",
The graph of "$y = -\frac{25x}{24}$" lies between the graphs of "$y = -x$" and "$x = 0$".

6. There are many lines through the origin. Where a line containing the origin lies with respect to the axes depends upon the coefficient of x in its open sentence. When the coefficient of x is positive, the line lies in quadrants I and III. When the coefficient of x is negative, the line lies in quadrants II and IV. When the absolute value of the coefficient is less than 1, the line lies between "$y = -x$" and "$y = 0$", or between "$y = x$" and "$y = 0$". When the absolute value of the coefficient is greater than 1, and line lies between "$y = -x$" and "$x = 0$" or between "$y = x$" and "$x = 0$".

7. Graphs of equations of the form "$y = kx$", where k is a real number are lines through the origin. When k is positive, the graph lies in quadrants I and III. When k is negative, the graph lies in quadrants II and IV. When k is between 0 and 1, the graph lies between the graphs of "$y = x$" and "$y = 0$". When $k > 1$, the graph lies between the graphs of "$y = x$" and "$x = 0$". When $k < -1$, the graph lies between the graphs of "$y = -x$" and "$x = 0$". When $|k| > 1$, the graph lies between the graphs of "$y = x$" and "$x = 0$",

[page 425]

or between the graphs of "$y = -x$" and "$x = 0$". When $|k| < 1$, the graph lies between the graphs of "$y = 0$" and "$y = x$", or "$y = -x$" and "$y = 0$". When k is 0, the graph is the x-axis.

Page 426. To find the ordinates of points for the third open sentence, "$y = \frac{2}{3}x - 3$", we subtract 3 from the ordinate of each point in the graph of the first. The coordinates of the points at which lines (a), (b), and (c) intersect the vertical axis are (0,0), (0,4) and (0,-3) respectively. The ordinate of each pair is the same number as that added to the term "$\frac{2}{3}x$" in the corresponding open sentence.

Page 426. The graph of "$y = \frac{2}{3}x + 4$" could be obtained by moving the graph of "$y = \frac{2}{3}x$" up 4 units.

The graph of "$y = \frac{2}{3}x - 3$" could be obtained by moving the graph of "$y = \frac{2}{3}x$" down 3 units.

The open sentences are:

$$y = \frac{2}{3}x + 6$$

$$y = \frac{2}{3}x - 6$$

Their graphs are shown in Figure 2.

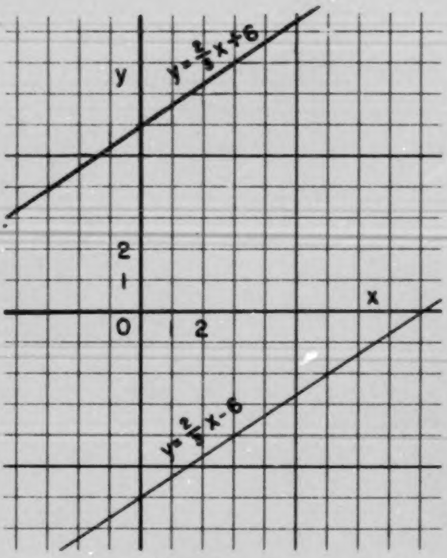


Figure 2.

The slope is negative when the line descends in going from left to right. The slope is 0 when the line is parallel to the x-axis. The line "$x = 2$" does not have a slope, because we cannot write "$x = 2$" in y-form. Thus, every non-vertical line has a slope, and only vertical lines do not have slopes.

Page 428. If we use as the first number in the numerator and denominator the ordinate and abscissa, respectively, of the point (2,2), the ratio is: $\frac{2-7}{2-4}$, or $\frac{5}{2}$. Thus we get the same value for the ratio regardless of which ordered pair we use first.

The slope of the line which contains the points (6,5) and (-2,-3) is $\frac{5-(-3)}{6-(-2)}$ or 1. The slope of the line which contains the points (2,7) and (7,3) is $\frac{3-7}{7-2}$ or $-\frac{4}{5}$.

Page 430. We have, essentially, a choice between two possible definitions of the slope of a line: the coefficient of x in the y-form of the line; the ratio of the vertical change to the horizontal change from one point to another on the (non-vertical) line. In a course in analytic geometry, in which a line is given a geometric meaning, the second of these would be taken as a definition and the first proved as a theorem. Here we have tacitly defined a line in terms of its equation, and it is natural to take the first as a definition and prove the second. Your better students will prefer to replace the wording of Theorem 14-3 by the more precise symbolism: If (a,b) and (c,d) are distinct points on a non-vertical line L, then the slope of L is

$$\frac{b-d}{a-c}.$$

Answers to Problem Set 14-3b; pages 430-431:

1. (a) $\frac{2-(-3)}{6-(-7)} = \frac{5}{13}$

(b) $\frac{3-3}{8-(-7)} = \frac{0}{15} = 0$

(c) $\frac{-1-6}{-4-8} = \frac{-7}{-12} = \frac{7}{12}$

(d) $\frac{10-(-12)}{-8-3} = \frac{22}{-11} = -2$

(e) $\frac{-2-11}{-1-4} = \frac{-13}{-5} = \frac{13}{5}$

(f) $\frac{0-5}{6-6} = \frac{-5}{0}$ - no slope

(g) $\frac{-2-0}{-6-0} = \frac{-2}{-6} = \frac{1}{3}$

(h) $\frac{4-0}{-7-0} = \frac{4}{-7} = -\frac{4}{7}$

Page 432. The equation of a line parallel to the line in Figure 10 of the text, but which contains the point (0,6) is "$y = -\frac{4}{3}x + 6$".

Answers to Problem Set 14-3c; page 433:

1. $y = \frac{4}{3}x + 6$

2. $y = \frac{4}{3}x - 12$

3. $-\frac{4}{3}$

4. $-\frac{2}{3}$

5. $y = -\frac{5}{6}x - 3$

6. Slope is: $\frac{11 - 4}{4 - 2} = \frac{7}{2}$; equation is: "$y = \frac{7}{2}x - 3$"

   check: $11 = \frac{7}{2}(4) - 3$

   $4 = \frac{7}{2}(2) - 3$

7. Slope is: $\frac{6 - (-4)}{5 - (-5)} = \frac{10}{10} = 1$; equation appears to be "$y = x$" but (5,6) and (-5,-4) are not on this line. There is no line satisfying these conditions.

Page 433. If the slope of the line had been $\frac{2}{3}$, we would have chosen points with respect to (0,6) by counting three units to the right and two units up, or three units to the left and two units down. Another point would be six units to the right and four units up, and so on.

Recall to the students that vertical lines have no slope, and such lines have equations "$x = k$", where k is a number. Thus, the equation of the line containing (-3,4) which has no slope is $x = -3$.

Answers to Problem Set 14-3d; pages 435-439:

1. (a) Since the y-intercept appears to be (0,2), on the graph, the equation of the line is:

$$y = \frac{5}{6}x + 2$$

(b) The line containing (-6,-3) which has no slope has equation:

$$x = -6.$$

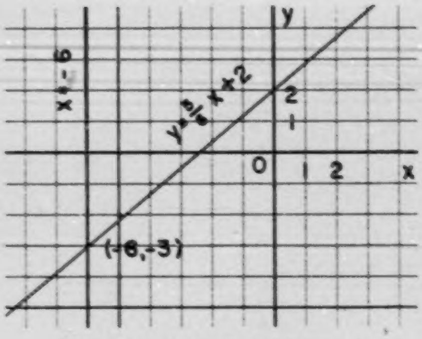


Figure for Problem 1.

2. Call attention to the distinction between the slopes of (c) and (e). For (c) the slope is 0 and the equation is "y = 4". For (e) there is no slope, since the denominator of the fraction form of the slope is 3 - 3 or 0.

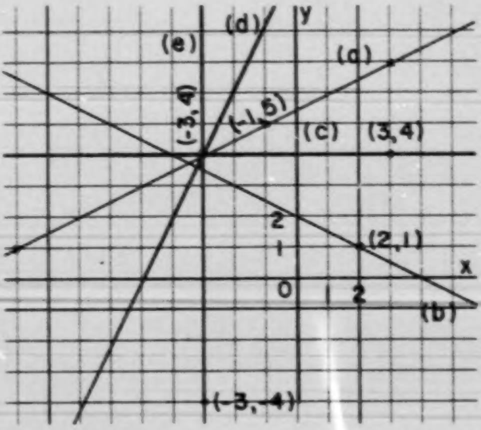


Figure for Problem 2.

3. The slope of the line containing (1,-1) and (3,3) is

$$\frac{3 - (-1)}{3 - 1} \text{ or } 2$$

The slope of the line containing (1,-1) and (-3,-4) is

$$\frac{-9 - (-1)}{-3 - 1} \text{ or } 2$$

Hence, the point (-3,-9) is on the line containing (1,-1) and (3,3).

4. (a) All of the lines pass through the origin, or have the point (0,0) in common.

   (b) All of the lines have the same slope, $\frac{1}{2}$.

   (c) All of the lines have the same y-intercept number, -3.

   (d) The lines have the same slope, $-\frac{1}{2}$. Moreover, the two whose open sentences are "$\frac{1}{2}x + y = 3$" and "$2x + 4y = 12$" have the same y-intercept number, hence, they are the same line, and are parallel to the first one.

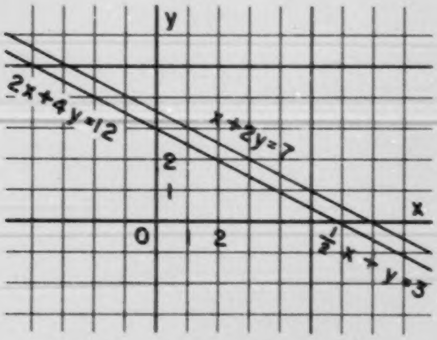


Figure for Problem 4(d).

5. (a) $3x + 4y = 12$ (b) $2x - 3y = 6$

$y = -\frac{3}{4}x + 3$ $y = \frac{2}{3}x - 2$

The y-intercept number of (a) is 3

The y-intercept number of (b) is -2. The slope of the first line is $-\frac{3}{4}$. The slope of the second line is $\frac{2}{3}$.

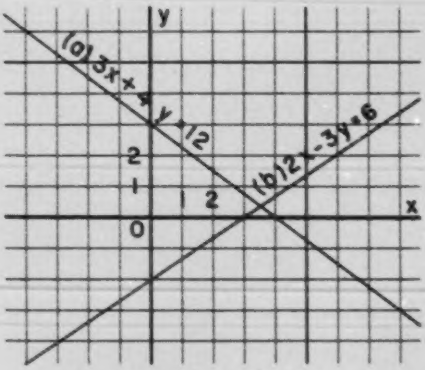


Figure for Problem 5.

6. (a) $y = 2x - 7$

(b) $y = \frac{3}{4}x - 3$

(c) $y = -\frac{4}{3}x + 4$

(d) $y = \frac{1}{2}x - 2$

The graphs of these are lines, because each open sentence is of the form

$Ax + By + C = 0$

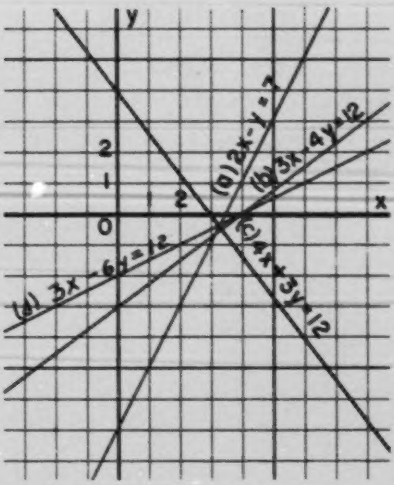


Figure for Problem 6.

7. (a) $y = \frac{2}{3}x$ (d) $y = -7x - 5$

(b) $y = \frac{3}{4}x - 2$ (e) $y = mx + b$

(c) $y = -2x + \frac{4}{3}$

The equation of every straight line can be put in the form $Ax + By + C = 0$. Every one of these can be put into the form $y = mx + b$ except those where $B = 0$. Neither the y-axis nor any lines parallel to it can be put in the form $y = mx + b$. The equation of the x-axis "$y = 0$" is of this form, in which m and b are both 0.

8. The slope is $\frac{4 - 0}{3 - 0}$, or $\frac{4}{3}$. The y-intercept is $(0,0)$. The equation is "$y = \frac{4}{3}x$".

9. Since the y-intercept number is 7, the line contains the point $(0,7)$. Since it also contains the point $(6,8)$, its slope could be written as $\frac{8 - 7}{6 - 0}$ or $\frac{1}{6}$. The equation of the line is "$y = \frac{1}{6}x + 7$".

10. The slope is $\frac{(-4) - 2}{3 - (-3)}$ or -1. The slope of the line containing $(-3,2)$ and (x,y) is $\frac{y - 2}{x - (-3)}$. The slope of the line containing $(3,-4)$ and (x,y) is $\frac{y - (-4)}{x - 3}$. Since -1 and $\frac{y - 2}{x - (-3)}$ are names for the same number,

$$\frac{y - 2}{x - (-3)} = -1, \quad \text{provided } x \neq -3$$

Then $y - 2 = (-1)(x + 3)$, by multiplying both sides by "$(x + 3)$", with the restriction that $x \neq -3$. Then

$$y = -x - 1.$$

Since -1 and $\frac{y - (-4)}{x - 3}$ are names for the same number

$$\frac{y - (-4)}{x - 3} = -1, \text{ provided } x \neq 3.$$

Then $y + 4 = (-1)(x - 3)$,

$y = -x - 1$.

11. (a) Slope is $\frac{2 - 3}{(-5) - 0}$ or $\frac{1}{5}$, and the y-intercept number is 3, so the equation is "$y = \frac{1}{5}x + 3$".

(b) Slope is $\frac{(-4) - 8}{0 - 5}$ or $\frac{12}{5}$, and y-intercept number is -4, so the equation is "$y = \frac{12}{5}x - 4$".

(c) Slope is $\frac{-7 - (-2)}{-3 - 0}$ or $\frac{5}{3}$, and the y-intercept number is -2, so the equation is "$y = \frac{5}{3}x - 2$".

(d) Slope is $\frac{6 - (-2)}{0 - 5}$ or $-\frac{8}{5}$, and the y-intercept number is 6, so the equation is "$y = -\frac{8}{5}x + 6$"

(e) Two expressions for the slope are

$\frac{y - 3}{x - (-3)}$ and $\frac{0 - 3}{6 - (-3)} = -\frac{1}{3}$, if $x \neq -3$.

Then $\frac{y - 3}{x + 3} = -\frac{1}{3}$,

an $y - 3 = -\frac{1}{3}(x + 3)$.

(f) Two expressions for the slope are

$\frac{y - 3}{x - (-3)}$ and $\frac{3 - 3}{-5 - (-3)} = 0$, if $x \neq -3$

then, $\frac{y - 3}{x + 3} = 0$,

and $y - 3 = 0$.

(g) The slope is $\frac{5 - 3}{-3 - (-3)}$ or $\frac{2}{0}$. But $\frac{2}{0}$ is not a number. Hence, the line has no slope. The only lines that have no slope are vertical lines. The vertical line through the point $(-3,3)$ has the equation "$x + 3 = 0$".

(h) Two expressions for the slope are

$\frac{y - 2}{x - 4}$ and $\frac{2 - 1}{4 - (-3)} = \frac{1}{7}$, if $x \neq 4$.

Then,

$$\frac{y - 2}{x - 4} = \frac{1}{7},$$

$$y - 2 = \frac{1}{7}(x - 4).$$

12.

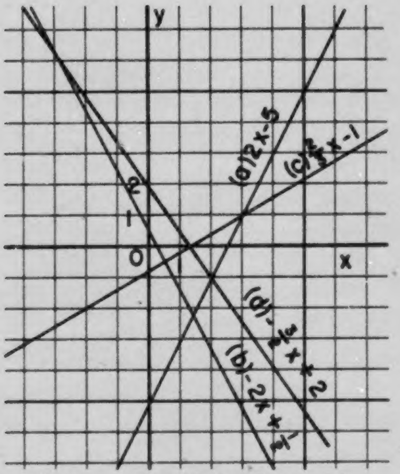


Figure for Problem 12.

*13. (a) $2w + 2(w + 3)$. This is a linear expression in w, since by using the distributive and associative properties it becomes "$4w + 6$".

(b) $w(w + 3)$, or $w^2 + 3w$. This is not a linear expression in w.

*14. (a) πd. This expression is linear in d. If the diameter is doubled, the circumference is doubled; if the diameter is halved, the circumference is halved. The ration $\frac{c}{d}$ is equal to π; this ratio does not change when d is changed.

(b) $\frac{1}{4} \pi d^2$. (If the student is not familiar with this, develop it as a combination of the two familiar relations, "area is πr^2" and "d is $2r$," or "r is $\frac{1}{2} d$"). This expression is not linear in d, but it is linear in d^2. If A is the area, $\frac{A}{d} = \frac{1}{4} \pi d$; $\frac{A}{d^2} = \frac{1}{4} \pi$. The value of $\frac{A}{d}$ changes when the value of d is changed; the value of $\frac{A}{d^2}$ does not change when d is changed.

*15. (a) The circumference of a circle varies directly as the diameter. The constant of variation is π. The area of the circle does not vary directly as the diameter, but it does vary directly as the square of the diameter. The constant of variation is $\frac{1}{4} \pi$.

(b) In terms of the graph of a linear expression, the constant of variation indicates the slope of the line.

(c) If the constant of variation is negative, the value of the expression decreases as the value of the variable increases.

(d) The expression would have the form "$k\sqrt{x}$".

*16. The distance in miles would be "k t". This is a linear expression in t. The distance varies directly as the time. The constant of variation is the speed in miles per hour. If the automobile has traveled 25 miles at the end of 20 minutes (which is $\frac{1}{3}$ of an hour) we find the constant of variation, k as follows:

$$k\left(\frac{1}{3}\right) = 25$$

$$k = 75$$

*17.

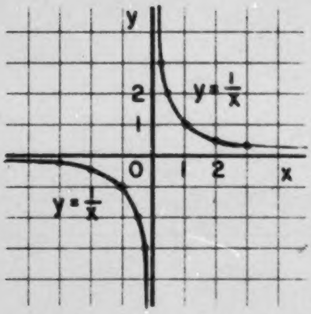


Figure for Problem 17(a).

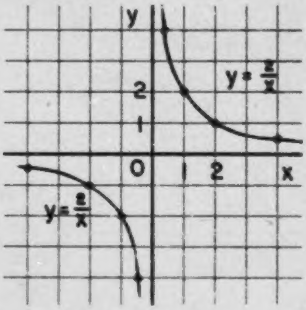


Figure for Problem 17(a).

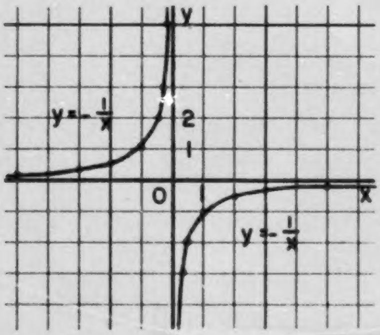


Figure for Problem 17(a).

(b) If the variable x is given increasing positive values, the values of $\frac{k}{x}$ decrease. If k is negative, then for increasing positive values of x the values of $\frac{k}{x}$ increase.

18. (a) $\frac{25}{w}$, where $w > 0$

(b) This is inverse variation, and the constant of variation is 25.

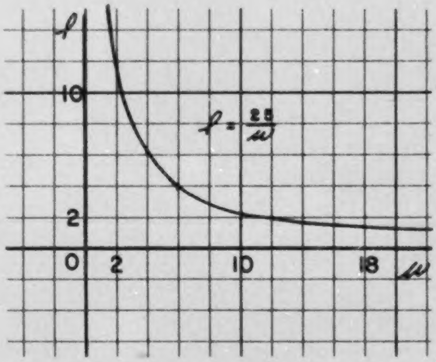


Figure for Problem 18.

Points used include:

w	$1\frac{1}{4}$	2	4	5	$6\frac{1}{4}$	$12\frac{1}{2}$	20
ℓ	20	$12\frac{1}{2}$	$6\frac{1}{4}$	5	4	2	$1\frac{1}{4}$

*14-4. Graphs of Open Sentences Involving Integers Only.

This section is included because it is hoped the student will realize that open sentences do not necessarily include all real numbers as possible members of their truth sets, and will recognize the corresponding situation so far as the graphs are concerned. Page 441. Each ordinate is one-third the corresponding abscissa. Hence, we get ordered pairs of integers only for abscissas which are multiples of 3. 1 and 2 are not multiples of 3, so they cannot be abscissas.

Answers to Problem Set 14-4; pages 444-447:

1. (a)

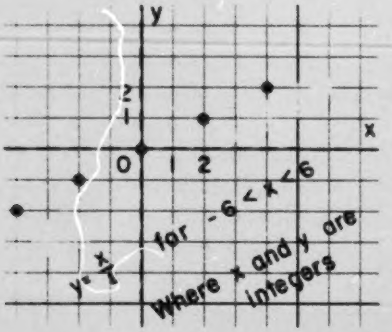


Figure for Problem 1(a).

(b)

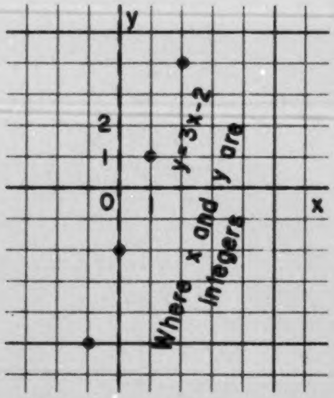


Figure for Problem 1(b).

(c)

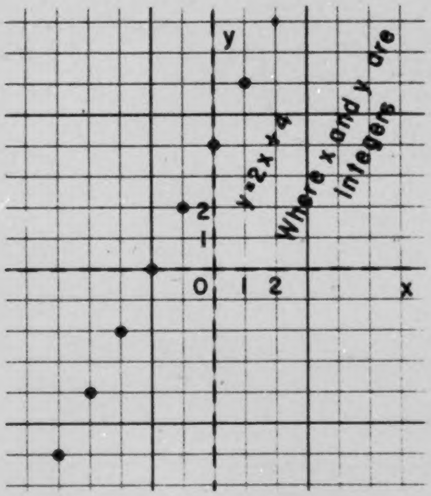


Figure for Problem 1(c).

2. (a)

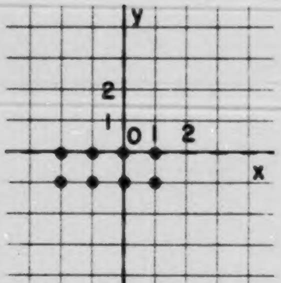


Figure for Problem 2(a).

(b)

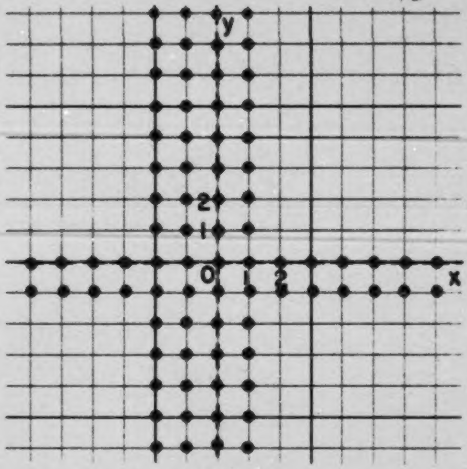


Figure for Problem 2(b).

(c)

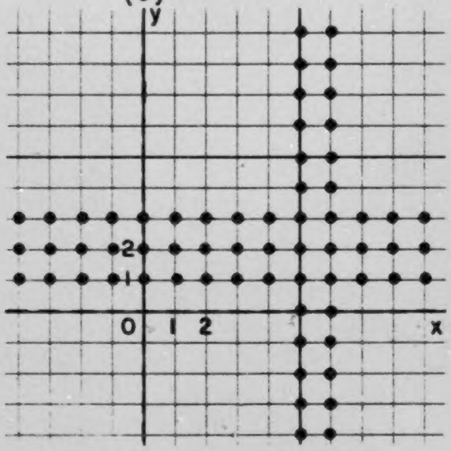


Figure for Problem 2(c).

(d)

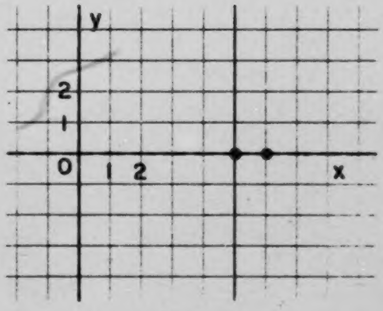


Figure for Problem 2(d).

3. $-2 < x < 0$ and $2 < y < 4$ where x and y are integers.

4. (a) $y = -3x + 1$, where x and y are integers.

   (b) $4 < x < 8$ and $-2 < y < 2$, where x and y are integers or $5 \leq x \leq 7$ and $-1 \leq y \leq 1$, where x and y are integers.

   (c) $y = -x -3$ for $-8 < x < -4$, where x and y are integers.

   (d) $x = -2$ and $2 < y < 7$, where x and y are integers.

(e) $x = -4$ or $-7 < y < -2$, where x and y are integers.

(f) $-2 < x < 2$ and $-4 < y < 3$, where x and y are integers.

(g) $x > -6$ and $y < 6$ and $y \geq x + 6$, where x and y are integers.

5. (a) (b)

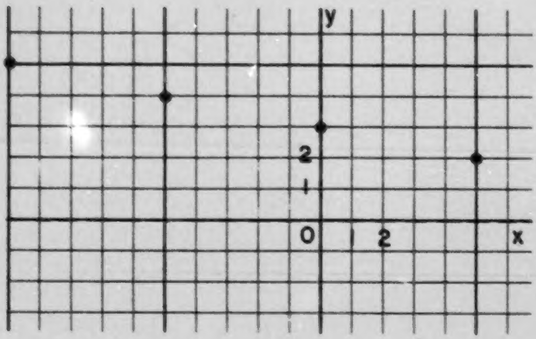


Figure for Problem 5(a).

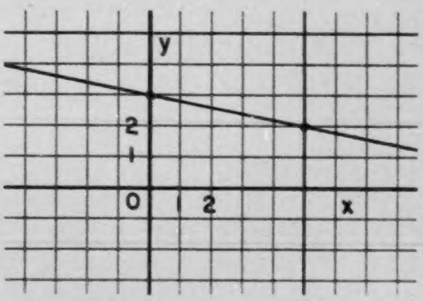


Figure for Problem 5(b).

The graph of (a) is a set of isolated points, while the graph of (b) is a straight line. Points on (b) but not on (a) include:

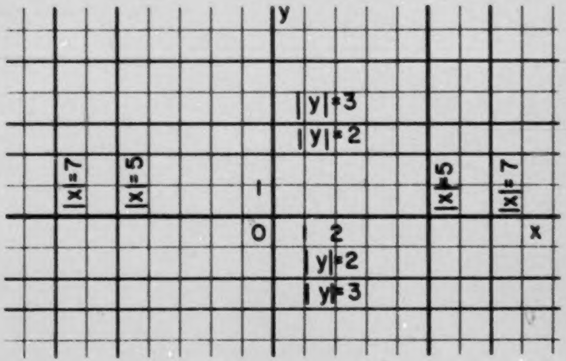
$(1,2\frac{4}{5})$, $(-2,3\frac{2}{5})$, etc.

The graph of (c) would have to be drawn as the graph of (b), since it would not be possible to indicate the "holes" for the irrational values. If x is rational, y is rational.

14-5. Graphs of Open Sentences Involving Absolute Value.

This section dealing with absolute value is valuable not only for the opportunity it provides for recall of work done with absolute value earlier in the course, but also for the opportunity it provides for examining what happens to a graph when certain changes are made in its equation.

Page 448. The graphs of $|x| = 5$ and $|x| = 7$ are pairs of vertical lines, and the graphs of $|y| = 2$ and $|y| = 3$ are pairs of horizontal lines. The graph of $|x| = k$ is a single line if and only if $k = 0$.



Answers to Problem Set 14-5a; pages 450-451:

1. (a)

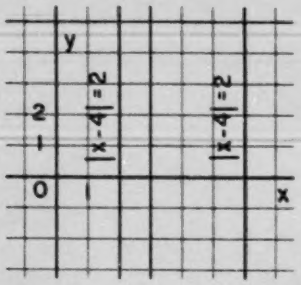

y
|x-4|=2
|x-4|=2
2
1
0
1
x

Figure for Problem 1(a).

(b)

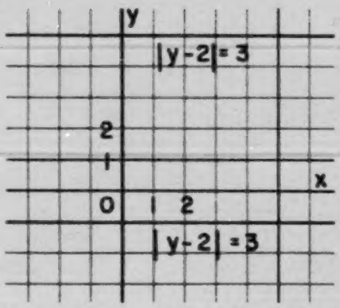

y
|y-2|=3
2
1
x
0
1
2
|y-2|=3

Figure for Problem 1(b).

(c)

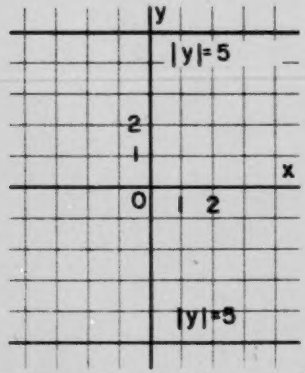

y
|y|=5
2
1
x
0
1
2
|y|=5

Figure for Problem 1(c).

(d)

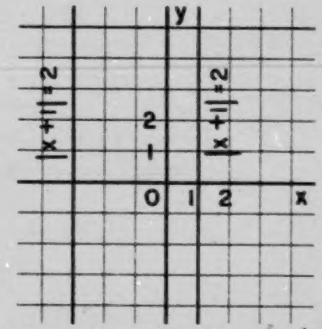

y
|x+1|=2
|x+1|=2
2
1
0
1
2
x

Figure for Problem 1(d).

(e)

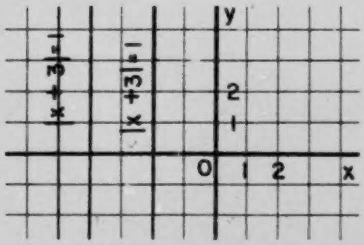

|x+3|=1
|x+3|=1
y
2
1
0
1
2
x

Figure for Problem 1(e).

(f)

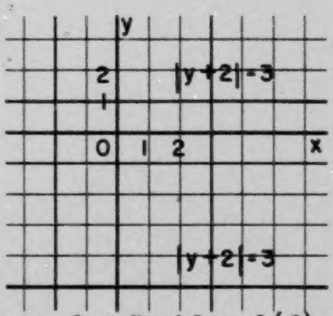

y
2
1
|y+2|=3
0
1
2
x
|y+2|=3

Figure for Problem 1(f).

2. (a)

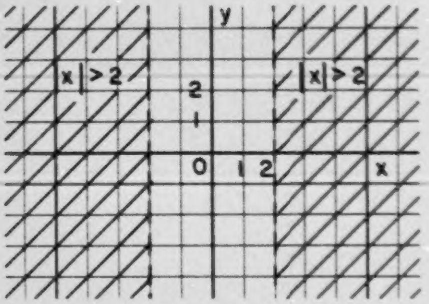


Figure for Problem 2(a).

(b)

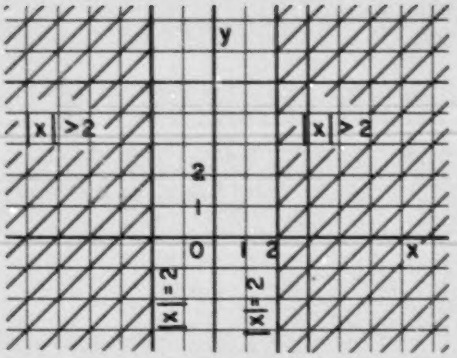


Figure for Problem 2(b).

(c)

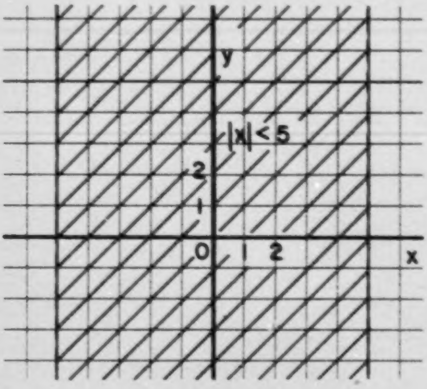


Figure for Problem 2(c).

3. (a)

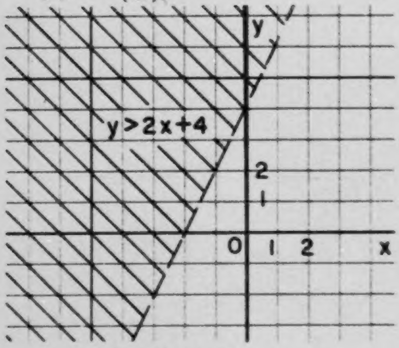


Figure for Problem 3(a).

(b)

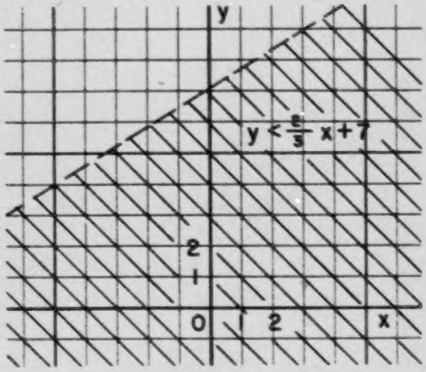


Figure for Problem 3(b).

3. (c)

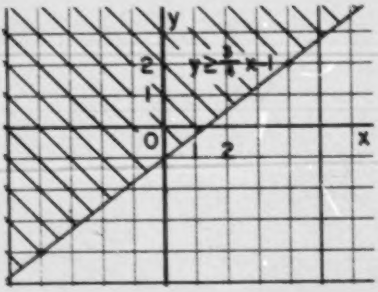


Figure for Problem 3(c).

(d)

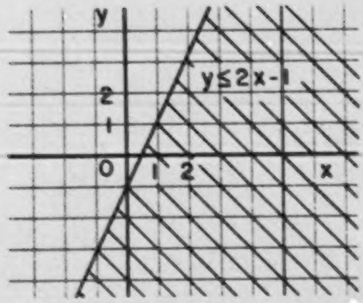


Figure for Problem 3(d).

4. (a)

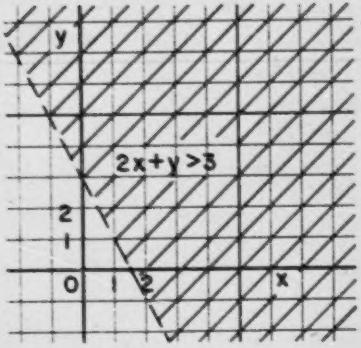


Figure for Problem 4(a).

(b)

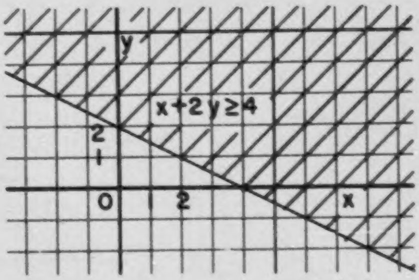


Figure for Problem 4(b).

(c)

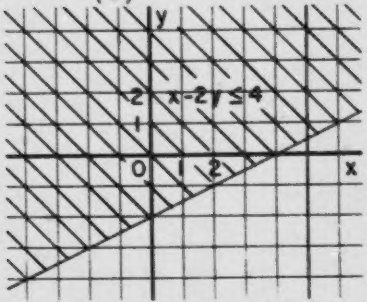


Figure for Problem 4(c).

(d)

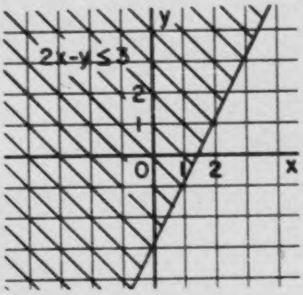


Figure for Problem 4(d).

5. (1) $y = 3x$

(2) $y = -x + 4$

(3) $y = \frac{1}{2}x - 6$

(4) $|x| = 6$

6.

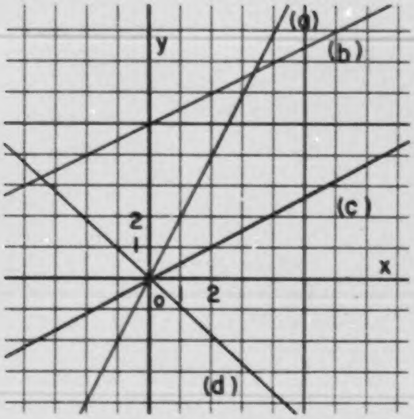


Figure for Problem 6.

Page 452. Whether x is positive or negative, the absolute value of x must be positive. So every value of y is positive for every value of x except 0. For $x = 0$, $y = 0$.

The two lines which are the graph of $y = |x|$ form a right angle, because each line forms one-half of a right angle with the line $y = 0$. A simple equation whose graph would be two lines which do not form a right angle would be "$y = 2\ |x|$" or "$y = -2\ |x|$", or any equation of the form "$y = k\ |x|$" where k is not 1.

Answers to Problem Set 14-5b; pages 453-454:

1. (a)

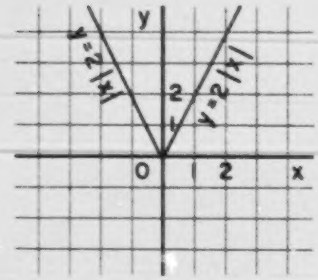


Figure for Problem 1(a).

(b)

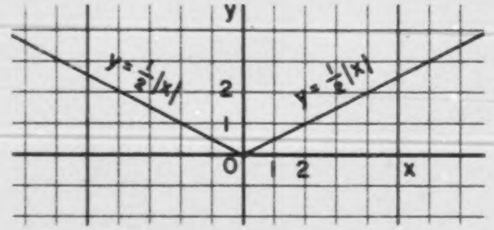


Figure for Problem 1(b).

(c)

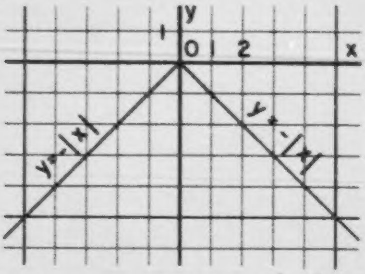


Figure for Problem 1(c).

(d)

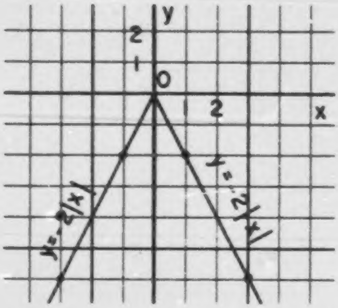


Figure for Problem 1(d).

(e)

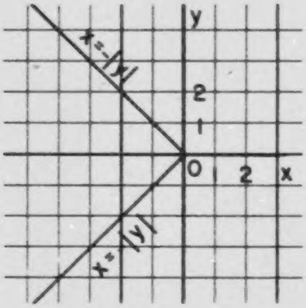


Figure for Problem 1(e).

(f)

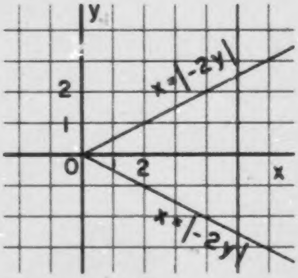


Figure for Problem 1(f).

2. (a)

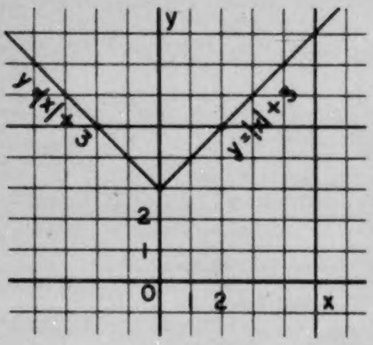


Figure for Problem 2(a).

(b)

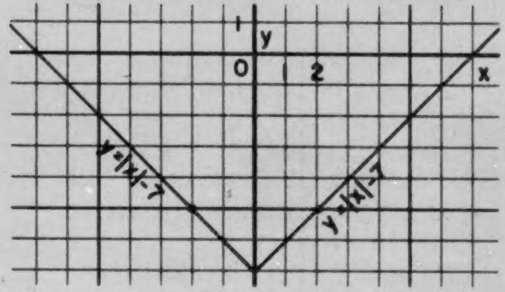


Figure for Problem 2(b).

(c)

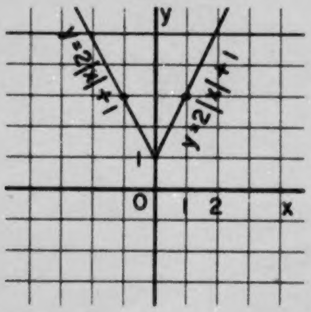


Figure for Problem 2(c).

(d)

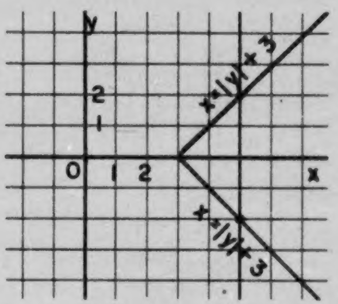


Figure for Problem 2(d).

(e)

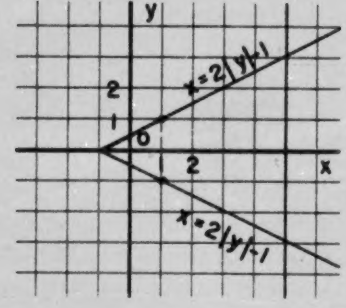


Figure for Problem 2(e).

(f)

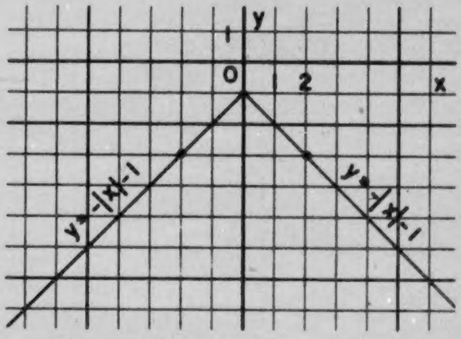


Figure for Problem 2(f).

[page 453]

3. (a)

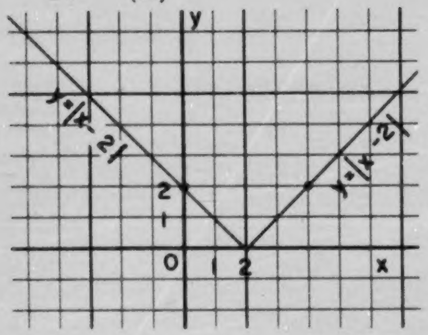


Figure for Problem 3(a).

(b)

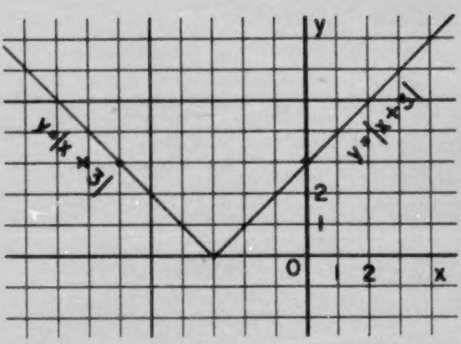


Figure for Problem 3(b).

(c)

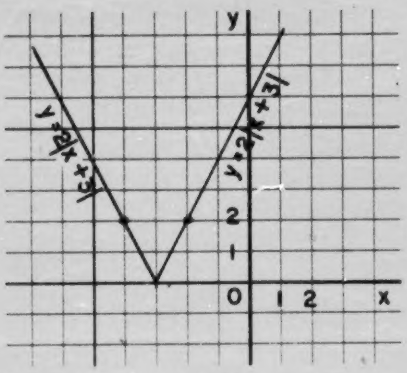


Figure for Problem 3(c).

(d)

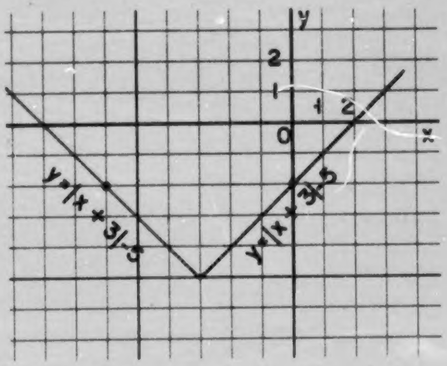


Figure for Problem 3(d).

(e)

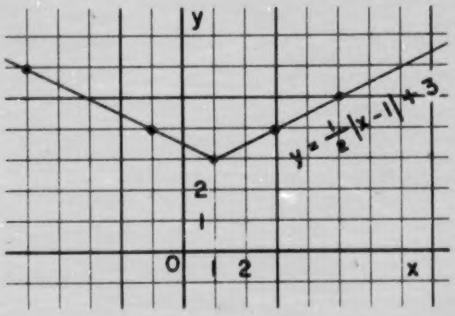


Figure for Problem 3(e).

[page 453]

4. Problem

1(c) The graph of "$y = -|x|$" can be obtained by rotating the graph of "$y = |x|$" $\frac{1}{2}$ revolution about the x-axis.

1(e) The graph of "$x = -|y|$" can be obtained by rotating the graph of "$x = |y|$" $\frac{1}{2}$ revolution about the y-axis.

2(a) The graph of "$y = |x| + 3$" can be obtained by sliding the graph of "$y = |x|$" up three units.

2(b) The graph of "$y = |x| -7$" can be obtained by sliding the graph of "$y = |x|$" down seven units.

2(d) The graph of "$x = |y| + 3$" can be obtained by sliding the graph of "$x = |y|$" to the right three units.

2(f) The graph of "$y = -|x| - 1$" can be obtained by rotating the graph of "$y = |x|$" about the x-axis and then sliding it down one unit.

3(a) The graph of "$y = |x - 2|$" can be obtained by sliding the graph of "$y = |x|$" to the right two units.

3(b) The graph of "$y = |x + 3|$" can be obtained by sliding the graph of "$y = |x|$" to the left three units.

3(d) The graph of "$y = |x + 3| - 5$" can be obtained by sliding the graph of "$y = |x|$" to the left three units and down five units.

*5. If $x = 6$, there are no possible values of y, since $|y| = -1$ and this is impossible. If $y = |2|$, there are two possible values for y: $y = -2$ and $y = 2$.

x	-5	-3	-3	-1	-1	0	0	1	1	3	3	5
$\|x\|$	5	3	3	1	1	0	0	1	1	3	3	5
$\|y\|$	0	2	2	4	4	5	5	4	4	2	2	0
y	0	2	-2	4	-4	5	-5	4	-4	2	-2	0

The graph shown is the graph of $|x| + |y| = 5$, as well as the graphs of the four open sentences:

$x + y = 5$, and $0 \leq x \leq 5$

or $x - y = 5$, and $0 \leq x \leq 5$

or $-x + y = 5$, and $-5 \leq x \leq 0$

or $-x - y = 5$, and $-5 < x < 0$

It was necessary in these to limit the values of $\underline{x}$ so that only the indicated segments of the lines would be included.

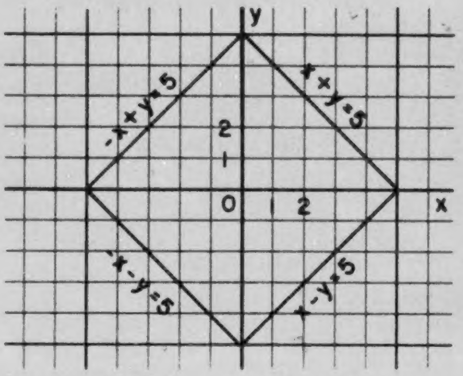


Figure for Problem 5.

*6. (a) Point out to the pupil the four open sentences implied here:

$x + y > 5$,

or $x - y > 5$,

or $-x + y > 5$,

or $-x - y > 5$.

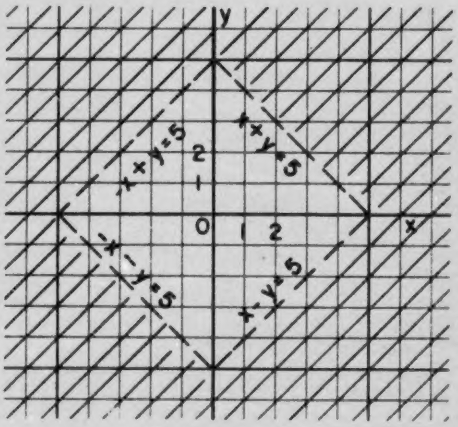


Figure for Problem 6(a).

The graphs of the corresponding equations: "$x + y = 5$ or $x - y = 5$ or etc." are then drawn with dotted lines. Now note that: $x + y > 5$ becomes $y > -x + 5$

$-x + y > 5$ becomes $y > x + 5$

So the area above each of the lines where "$x + y = 5$" and "$-x + y = 5$" is shaded.

Also: $x - y > 5$ becomes $y < x - 5$

$-x - y > 5$ becomes $y < -x - 5$

So the area below each of the lines where "$x - y = 5$" and "$-x - y = 5$" is shaded.

Therefore the graph of $|x| + |y| > 5$ is all of the plane outside the graph of $|x| + |y| = 5$.

(b) In the same line of reasoning

$|x| + |y| < 5$ implies:

$x + y < 5$,

and $x - y < 5$,

and $-x + y < 5$,

and $-x - y < 5$.

Hence, the graph is the area inside the graph of

$|x| + |y| = 5$.

Verify on the number line that "$|y| < k$" is equivalent to "$y < k$ and $-y < k$", whereas "$|y| > k$" is equivalent to "$y > k$ or $-y > k$".

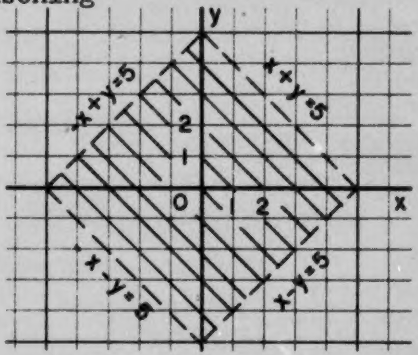


Figure for Problem 6(b).

(c) The graph is the same as that for (b), except that the lines are solid to indicate that the graph of

$$|x| + |y| = 5$$

is included, as well as the graph of $|x| + |y| < 5$.

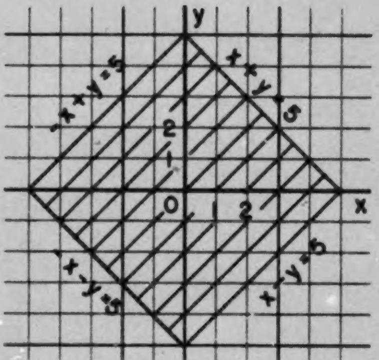

y
-x+y=5
x+y=5
2
1
0
1
2
x
-x-y=5
x-y=5

Figure for Problem 6(c).

(d) "$|x| + |y| \not< 5$" implies "$|x| + |y| = 5$ or $|x| + |y| > 5$", so the graph is the same as that for (a), except that the lines are solid.

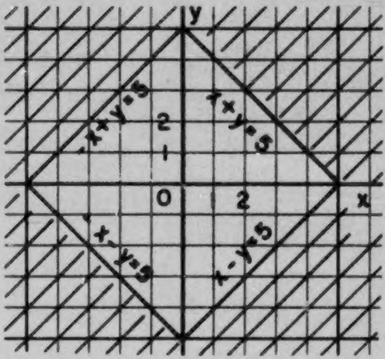

y
-x+y=5
x+y=5
2
1
0
2
x
-x-y=5
x-y=5

Figure for Problem 6(d).

*7.

x	-7	-7	-6	-6	-4	-4	-3	-2	1	3	4	4	7	7
|x|	7	7	6	6	4	4	3	2	1	3	4	4	7	7
|y|	4	4	3	3	1	1	0	-1	-2	0	1	1	4	4
y	4	-4	3	-3	1	-1	0	Impossible		0	1	-1	4	-4

The four open sentences whose graphs form the same figure are:

$x - y = 3$, and $x \geq 3$
$x + y = 3$, and $x \geq 3$
$-x + y = 3$, and $x \leq -3$
$-x - y = 3$, and $x \leq -3$

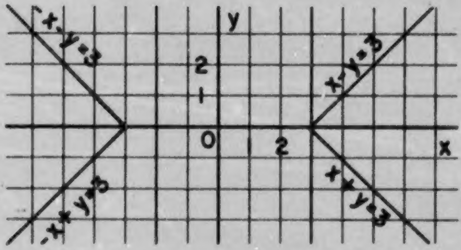

-x-y=3
y
x-y=3
2
1
0
1
2
x
-x+y=3
x+y=3

Figure for Problem 7.

Answers to Review Problems, pages 455-463:

1. (a) $y = 3x$

   (b) $y = -\frac{3}{2}x - 3$

   (c) $0 < x < 3$ and $0 < y < 5$, where x and y are integers. (or: $1 \leq x \leq 2$ and $1 \leq y \leq 4$, where x and y are integers.)

   (d) $x = 2$ and $-1 < y < 8$, where y is an integer (or: $x = 2$ and $0 \leq y \leq 7$, where y is an integer)

   (e) $y = 2\,|x|$

   (f) $y = |x| - 2$

   (g) $y = |2x - 4|$, i.e., $y = 2\,|x - 2|$

   (h) $|x| + |y| \leq 8$

   (i) $y \leq \frac{3}{2}x + 2$

   (j) $y \geq 2\,|x - 3|$

   (k) $y > -x + 4$

   (ℓ) $x < 6$ and $y > 0$ and $y \leq x$, where x and y are integers; or $x \leq 5$ and $y \geq 1$ and $y \leq x$, where x and y are integers.

2. (a) (b)

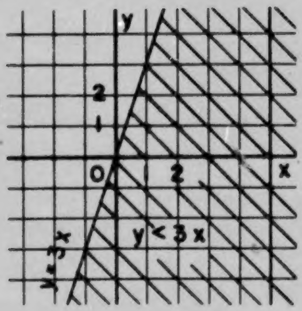


Figure for Problem 2(a).

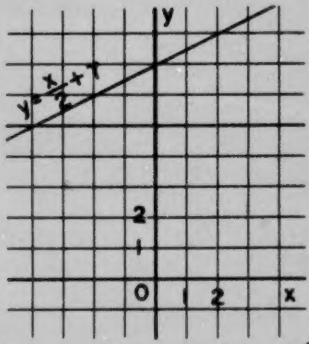


Figure for Problem 2(b).

(c)

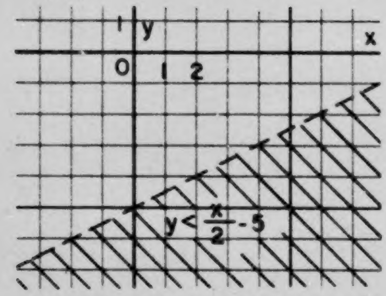


Figure for Problem 2(c).

(d)

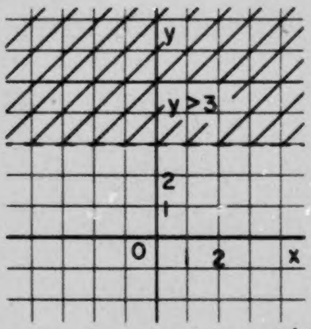


Figure for Problem 2(d).

(e)

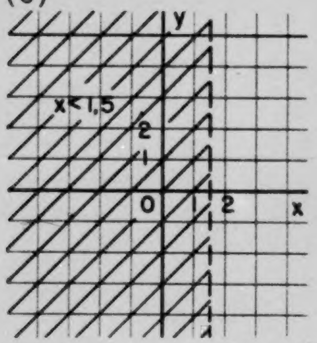


Figure for Problem 2(e).

(f)

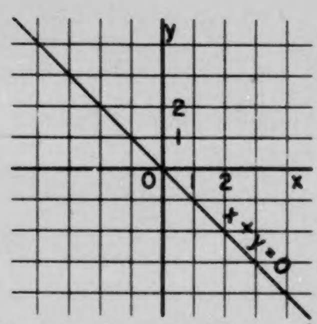


Figure for Problem 2(f).

(g) No graph is possible, since "|x| + |y|" must be positive. Therefore, the truth set of "|x| + |y| = -2" is $\emptyset$.

(h)

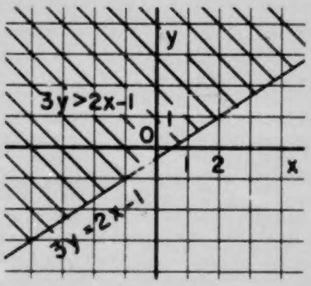


Figure for Problem 2(h).

(i)

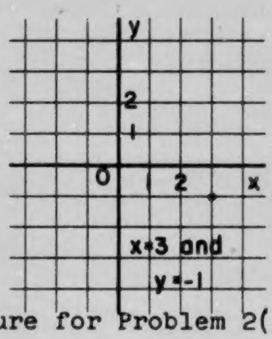


Figure for Problem 2(i).

(j)

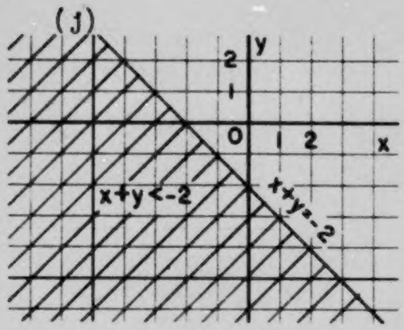

y
2
1
0
1 2
x
x+y<-2
x+y=-2

Figure for Problem 2(j).

(k)

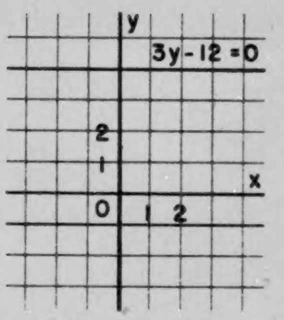
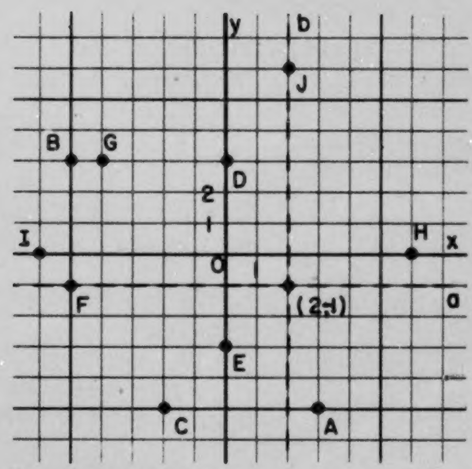

y
3y - 12 = 0
2
1
x
0
1 2

Figure for Problem 2(k).

*3.

	(a, b)
A	(1, -4)
B	(-7, 4)
C	(-4, -4)
D	(-2, 4)
E	(-2, -2)
F	(-7, 0)
G	(-6, 4)
H	(4, 1)
I	(-8, 1)
J	(0, 7)


y
b
J
B
G
D
2
1
I
H
x
O
F
(2,-1)
a
E
C
A

4. L_1: $y = -\frac{3}{2}x - 7$; $b = -\frac{3}{2}a$

L_2: $y = 2x - 5$; $b = 2a - 5$

L_3: $y = -\frac{1}{2}x$; $b = -\frac{1}{2}a + 5$

L_4: $y = -9$; $b = -5$

L_5: $|x - 5| = 1$; $|a - 7| = 1$

5. (a) a is negative

(b) b is positive

(c) P (a,-b); Q(-a,-b); R(-a,b)

6. (c,-d) is in quadrant II

(-c,d) is in quadrant IV

(-c,-d) is in quadrant I

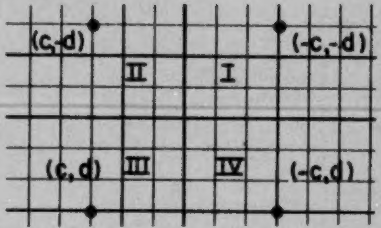


Figure for Problem 6.

7. (a) The graph of "$y = 3x + 4$" is rotated about the y-axis.

(b) The graph of "$y = 3x + 4$" is rotated about the x-axis.

(c) The graph is moved down 3 units.

(d) The graph is moved 2 units to the right.

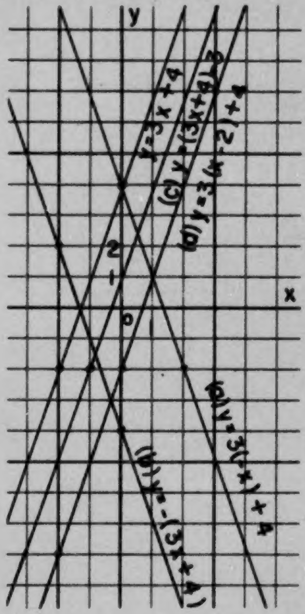


Figure for Problem 7(a) - 7(d).

[page 459-460]

*8. (a) $y = -2\,|x|$

(b) $y = 2\,|x - 3|$

(c) $y = 2\,|x + 2|$

(d) $y = 2\,|x| + 5$

(e) $y = 2\,|x - 2| - 4$

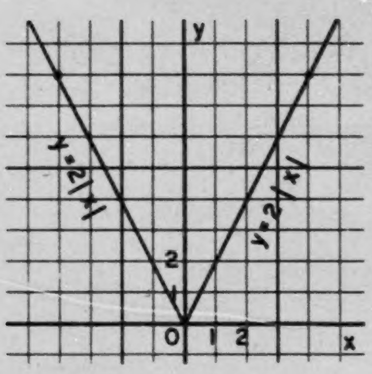


Figure for Problem 8.

9. (a) The same line was the graph of each equation. You could get the second equation by multiplying the members of the first equation by 3.

(b) The graphs would be the same straight line.

(c) The graphs of the two equations will be the same line if there is a number k such that:

$$D = kA, \quad E = kB, \quad \text{and} \quad F = kC.$$

If the graphs are the same line, then

$$\frac{A}{D} = \frac{B}{E} = \frac{C}{F}$$

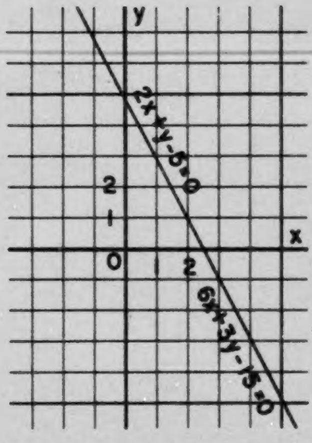


Figure for Problem 9.

[page 461]

10. (a) The graphs have the same slopes, and are parallel lines. The coefficients of x and y in the second equation are four times the coefficients of x and y in the first equation.

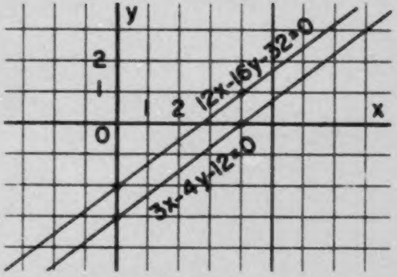


Figure for Problem 10.

(b) The graphs would be parallel lines, unless D = kC, in which case they would be the same line.

(c) The graphs will be parallel lines if there is a number k such that D = kA, E = kB, and F ≠ kC. If the graphs are parallel lines, then

$$\frac{A}{D} = \frac{B}{E} \neq \frac{C}{F}.$$

11. (a)

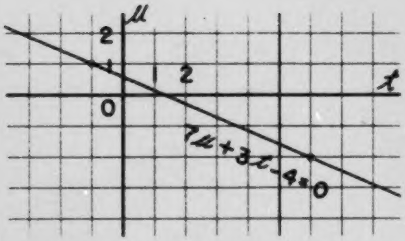


Figure for Problem 11(a).

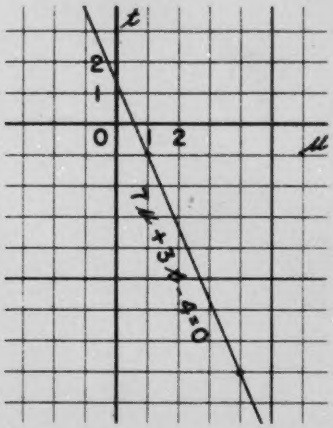


Figure for Problem 11(a).

(b)

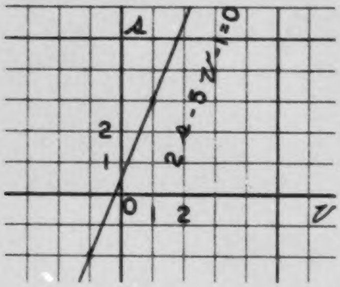


Figure for Problem 11(b).

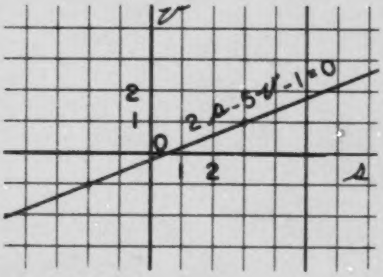


Figure for Problem 11(b).

In both cases, it is obvious that the graphs depend on the choices made for the first variable. We speak of sentences with two ordered variables because we are dealing with ordered pairs of real numbers, and unless the variables are also ordered we have no way of knowing which number of an ordered pair corresponds to which variable.

*12. Each point of the plane is moved to a point having the same abscissa as before, and the ordinate of the new point is the opposite of the ordinate of the original point. This amounts to rotating the points of the plane one-half revolution about the x-axis.

(a) (2,1) goes to (2,-1)
(2,-1) goes to (2,1)
$(-\frac{1}{2},2)$ goes to $(-\frac{1}{2},-2)$
(-2,-3) goes to (-2,3)
(3,0) goes to (3,0)
(-5,0) goes to (-5,0)
(0,5) goes to (0,-5)
(0,-5) goes to (0,5)

(b) (2,-1) goes to (2,1)
(2,1) goes to (2,-1)
$(-\frac{1}{2},-2)$ goes to $(-\frac{1}{2},2)$
(-2,3) goes to (-2,-3)
(3,0) goes to (3,0)
(-5,0) goes to (-5,0)
(0,-5) goes to (0,-5)
(0,5) goes to (0,-5)

(c) (a,-b) goes to (a,b)

(d) (-a,b) goes to (-a,-b)

(e) (a,-b) goes to (a,b)

(f) All points on the x-axis go to themselves.

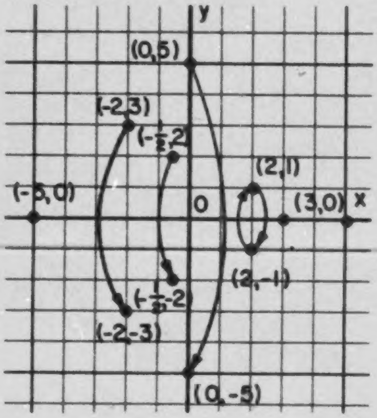


Figure for Problem 12.

*13. The points of the plane move up two units and to the left three units.

(a) (1,1) goes to (-2,3)
(-1,-1) goes to (-4,1)
(-2,2) goes to (-5,4)
(0,-3) goes to (-3,-1)
(3,0) goes to (0,2)

(b) (4,-1) goes to (1,1)
(2,-3) goes to (-1,-1)
(1,0) goes to (-2,2)
(3,-5) goes to (0,-3)
(6,-2) goes to (3,0)

(c) (a,b-2) goes to (a-3,b)

(d) (-a+3,-b-2) goes to (-a,-b)

(e) No point goes to itself

(f) Moving (a,b) to (a,b-2) has the effect of sliding the points of the plane down two units.

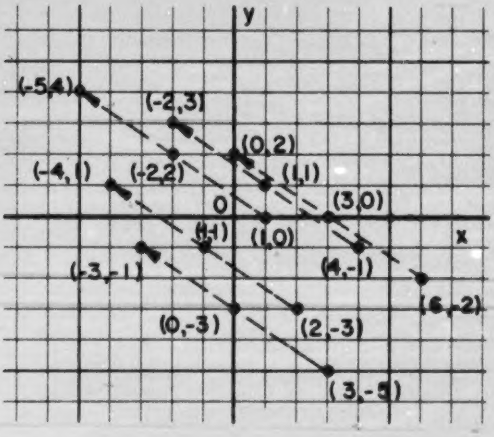

y
x
O
(-5,4)
(-2,3)
(0,2)
(-4,1)
(-2,2)
(1,1)
(3,0)
(-1,-1)
(1,0)
(4,-1)
(-3,-1)
(6,-2)
(0,-3)
(2,-3)
(3,-5)

Figure for Problem 13.

Suggested Test Items

1. Consider the equation $2x - 5y + 6 = 0$

   (a) Write the equation in the y-form.

   (b) What is the slope of the line with this equation?

   (c) What is the y-intercept of this line?

   (d) Draw the graph of the equation.

   (e) What is an equation of the line parallel to the given line and with y-intercept number -2?

2. What is the value of a such that the line with equation $3x + 2y - 6 = 0$ contains the point $(a,3)$?

3. What is the value of b such that the point $(2,-3)$ is on the line with equation $2x - by = 3$?

4. Determine the slope of each of the lines whose equations are:

   (a) $y - 3 = 0$

   (b) $x = 2y - 2$

   (c) $-x + 1 = 0$

5. Given the equation $x^2 - 1 - y = 0$ and the ordered pairs $(0,2)$, $(-3,8)$, $(1,1)$, $(-2,4)$, $(0,-1)$, $(0,1)$, $(-2,3)$, Which of the given ordered pairs are elements of the truth set of the given equation?

6. Give a reason why or why not the equation in Problem 5 has a graph which is a line.

7. Draw the graphs of each of the following with reference to a different set of axis.

   (a) $2x + y + 5 = 0$

   (b) $y = \frac{3}{2}x - 2$

   (c) $2x - 1 = 0$

   (d) $|y| = x$

   (f) $2y + 3 = 0$

   (g) $y - x > 0$

   (h) $x - 2y > 0$

   (i) $x + 2 = y$ or $x = y$

(e) $x = 2y + 3$ (j) $x + 2 = y$ and $x = y$

8. Draw a line such that the coordinates of its points are the ordered pairs for which each ordinate is twice the opposite of the abscissa. What is the equation of this line?

9. From one point to another on a line the horizontal change is -3 units and the vertical change is 6 units. What is the slope of the line?

10. If the line described in Problem 9 contains the point $(0,-\frac{1}{2})$, what is an equation of the line?

11. Is the point $(-1,4)$ on the line containing the points $(-6,7)$ and $(9,-3)$? Give a reason for your answer.

12. With respect to separate sets of axes, draw the graphs of

    (a) $x \geq -4$ and $x < -1$, with x and y integers.

    (b) $x \geq -4$ and $x < -1$, with x and y real numbers.

13. With respect to separate sets of axes, draw the graphs of

    (a) $|x - 1| = 3$ (b) $|x - 1| < 3$

14. With respect to separate sets of axes, draw the graph of

    (a) $y = |x| + 1$ (c) $x + 2y > 4$

    (b) $y = |x + 1|$ (d) $|x + 2y| > 4$

Chapter 15

SYSTEMS OF EQUATIONS AND INEQUALITIES

The system of linear equations

$$\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0, \end{cases}$$

that is, the conjunction

$$Ax + By + C = 0 \ \underline{\text{and}}\ Dx + Ey + F = 0,$$

arises in many contexts where two variables have two conditions placed on them simultaneously. This probably explains why such a system is often called a "system of simultaneous equations".

We want the students to continue extending their ideas about sentences, truth sets, and graphs. Thus, such a system is another example of a sentence in two variables, and we again face the problem of describing its truth set and drawing its graph. As before, we solve this sentence by obtaining an equivalent sentence whose truth set is obvious. Here we are aided by the intuitive geometry of lines. Two lines either intersect in exactly one point or they are parallel. If the lines given by the system intersect, then the point of intersection must have coordinates satisfying both equations of the system, and this ordered pair is the solution of the sentence. Thus, the problem is one of finding two lines through this point of intersection whose equations are the most simple, namely, a vertical line and a horizontal line. All methods of solving such systems are actually procedures for finding these two lines.

Page 466. Our symbolism

$$\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0 \end{cases}$$

is just another way of writing the open sentence

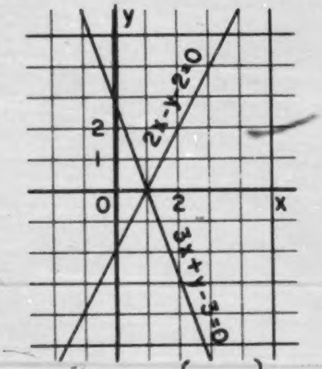
$$Ax + By + C = 0 \ \underline{\text{and}}\ Dx + Ey + F = 0.$$

The students must not interpret the symbolism as the disjunction "Ax + By + C = 0 or Dx + Ey + F = 0".

In Problems 1(a) through 1(e) of Problem Set 15-1a, the students are to estimate the coordinates of the points of intersection from carefully drawn graphs and then verify that their guesses (estimates) are good ones.

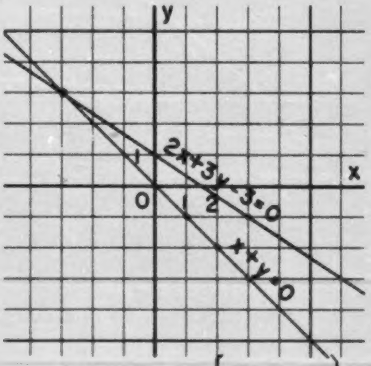
Answers to Problem Set 15-1a; page 467:

1(a)



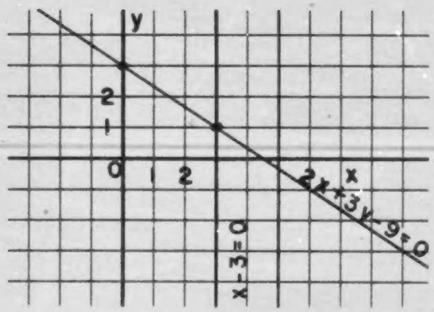
Truth set: $\{(1,0)\}$

1(b)



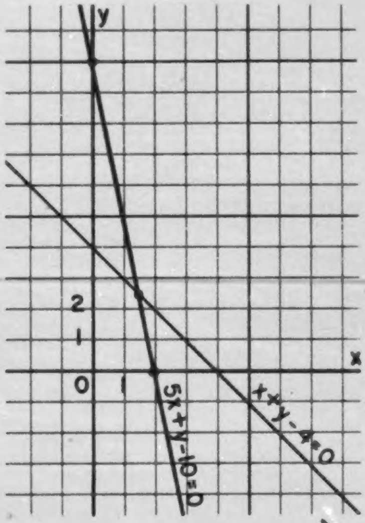
Truth set: $\{(-3,3)\}$

1(c)



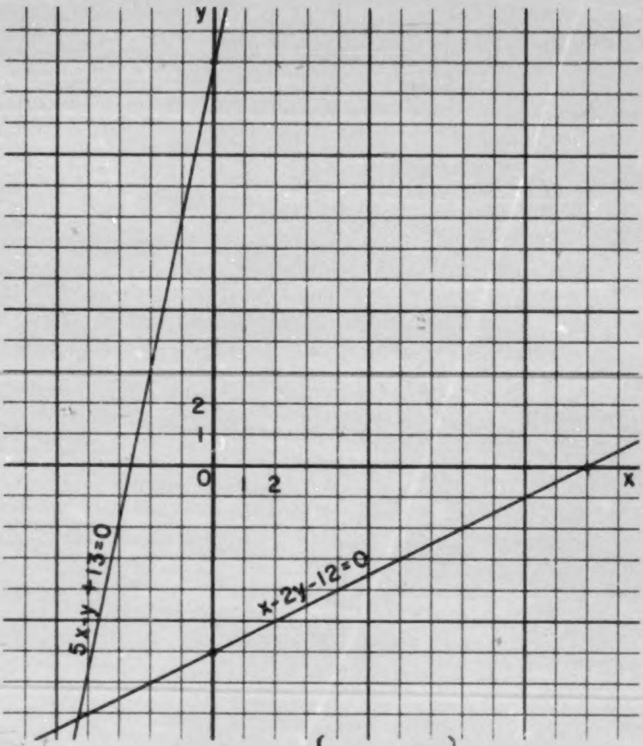
Truth set: $\{(3,1)\}$

1(d)



Truth set: $\{(\frac{3}{2}, \frac{5}{2})\}$

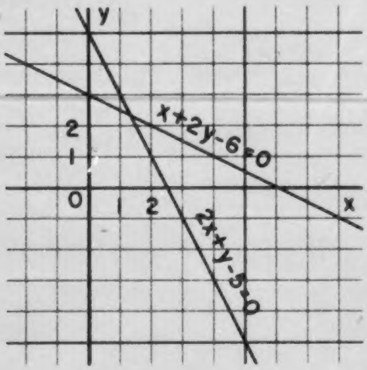
1 (e)



Truth set: $\left\{\left(-\frac{38}{9}, -\frac{73}{9}\right)\right\}$

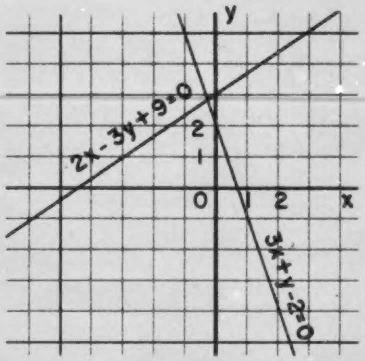
The student's estimates might be $\{(-4.2, -8.1)\}$ or $\{(-4, -8)\}$. Do not try to bring out the exact values at this time. This problem will be used as a basis of discussion in the next pages.

2 (a)



Truth set: All points of both lines.

2 (b)



Truth set: All points of both lines.

Page 467. Other compound sentences with truth set $\{(-1,3)\}$ are

$$\begin{array}{lcl} x + 1 = 0 & \text{and} & y - 3 = 0, \\ x + 1 = 0 & \text{and} & 2x + y - 1 = 0, \\ x + 2y - 5 = 0 & \text{and} & y - 3 = 0, \end{array}$$

Pages 468-472. Our purpose here is to derive a method for solving a system of equations

$$\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0 \end{cases} \quad (A \neq 0 \text{ or } B \neq 0;\ D \neq 0 \text{ or } E \neq 0).$$

on the assumption that this system has exactly one ordered pair of real numbers as its solution. We think of the individual clauses of the system as equations of lines and try to find a horizontal line and a vertical line which pass through the point common to this first pair of lines. If these new lines have equations "$y = d$" and "$x = c$", then (c,d) is the solution of our system.

The lines of the system $\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0 \end{cases}$ have exactly one point in common if and only if they are neither parallel nor coincident. We know that two lines are parallel or coincident if and only if they are both vertical or both have the same slope. Putting these statements together we can say that the lines of the above system have exactly one point in common if and only if

(i) $B \neq 0$ or $E \neq 0$ (the lines are not both vertical)

and (ii) $-\frac{A}{B} \neq -\frac{D}{E}$ (if non-vertical, the lines are not parallel).

A very astute student may inquire how these conditions affect the method for solving linear systems which is given in the text. He may wonder why, for example, we are always able to select proper multipliers a and b which yield equations of horizontal and vertical lines through the point common to the given lines, if there is exactly one point. An explanation, which is certainly not intended for ordinary class discussion, depends on the following:

[pages 467-472]

Theorem. The lines of the system

$$\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0 \end{cases}$$

are parallel or coincident if and only if there exist real numbers a and b, not both zero, such that

$$aA + bD = 0 \text{ and } aB + bE = 0.$$

Discussion: A rewording of one part of this Theorem is: If the lines are neither parallel nor coincident, then there exist real numbers a and b, not both zero, such that

$$aA + bD \neq 0 \text{ or } aB + bE \neq 0.$$

(Notice that we have reworded part of the Theorem in the form of a contrapositive. That is, if statement A implies statement B, then the negation of B implies the negation of A.) As we shall see, the fact that either $aA + bD \neq 0$ or $aB + bE \neq 0$ will guarantee the success of the method used in the text.

Proof: Let us prove first that if the lines are parallel or coincident, then there exist numbers a and b, not both zero, such that $aA + bD = 0$ and $aB + bE = 0$. There are three cases. Either the lines are vertical, or they are horizontal or neither of these. If they are vertical, then

$$B = 0, \quad E = 0, \quad A \neq 0, \quad D \neq 0,$$

and we may choose $a = -D$ and $b = A$ such that

$$aA + bD = (-D)A + AD = 0 \quad \text{and} \quad aB + bE = a \cdot 0 + b \cdot 0 = 0.$$

If they are horizontal, then

$$A = 0, \quad D = 0, \quad B \neq 0, \quad E \neq 0$$

and we may choose $a = -E$ and $b = B$ so that

$$aA + bD = a \cdot 0 + b \cdot 0 = 0 \quad \text{and} \quad aB + bE = -EB + BE = 0.$$

If the lines are neither vertical nor horizontal (and are parallel), then

$$-\frac{A}{B} = -\frac{D}{E},\ A \neq 0,\quad B \neq 0,\quad C \neq 0,\quad D \neq 0,$$

that is,

$$\frac{A}{D} = \frac{B}{E}$$

In this case, we may choose $a = -1$ and $b = \frac{A}{D} = \frac{B}{E}$, giving

$$aA + bD = -A + \frac{A}{D}\cdot D = 0 \quad \text{and} \quad aB + bE = -B + \frac{B}{E}\cdot E = 0.$$

Conversely, we prove that if there exist real numbers a and b, not both zero, such that $aA + bD = 0$ and $aB + bE = 0$, then the lines are parallel or coincident. There are two cases: either $a \neq 0$ or $a = 0$. if $a \neq 0$, we may write the conditions in the form

$$A = \frac{b}{a}D \quad \text{and} \quad B = \frac{b}{a}E.$$

Since the equation $Ax + By + C = 0$ cannot have $A = B = 0$, it follows that $b \neq 0$. Hence, the equation $Ax + By + C = 0$ is equivalent to

$$\frac{b}{a}Dx + \frac{b}{a}Ey + C = 0\ ,$$

that is, to

$$Dx + Ey + \frac{aC}{b} = 0.$$

This is the equation of a line parallel to or coincident with the line whose equation is $Dx + Ey + C = 0$. If $a = 0$, then $bD = 0$ and $bE = 0$. Since $b \neq 0$, it follows that $D = 0$ and $E = 0$ and the lines are both vertical, hence, parallel or coincident. This completes the proof.

The theorem just proved allows us to justify the method of the text. If the lines $Ax + By + C = 0$ and $Dx + Ey + F = 0$ have exactly one point in common, we form the equation

$$a(Ax + By + C) + b(Dx + Ey + F) = 0$$

with a and b real numbers, at lest one of which is not 0. This new equation may be written

$$(aA + bD)x + (aB + bE)y + (aC + bF) = 0.$$

Since the lines of the original system are neither parallel nor coincident, we know that $aA + bD \neq 0$ or $aB + bE \neq 0$ and, hence, that this new equation is an equation of a line. Furthermore, if (c,d) is the solution of our original system, then $Ac + Bd + C = 0$ and $Dc + Ed + F = 0$, so that

$$a(Ac + Bd + C) + b(Dc + Ed + F) = a(0) + b(0) = 0;$$

in other words, the new line passes through the intersection of the lines of the original system.

All we have to do to get a horizontal line containing (c,d) is choose a and b so that at least one of them is not 0 and $aA + bD = 0$. If A and D are 0, the lines of the original system are both horizontal and, hence, either have no points in common or all points in common, contrary to our original assumption that (c,d) is the only point common to the two lines of the system. Thus, at least one of A and D is not 0, and we simply choose $a = D$ and $b = -A$.

Similarly, the choice $a = E$ and $b = -B$ gives us a vertical line on (c,d).

Your students will certainly not appreciate the argument we have just given. They will be able to recognize the situations where the lines of the system are neither parallel nor coincident and, so, will know whether or not the system has a unique solution. However, they should always check their "solution" in both equations of the system to see that their computations are correct.

The method given in the text has a geometric flavor: lines, points common to lines, etc. A purely algebraic method is given in Problem 2 of Problem Set 15-1b.

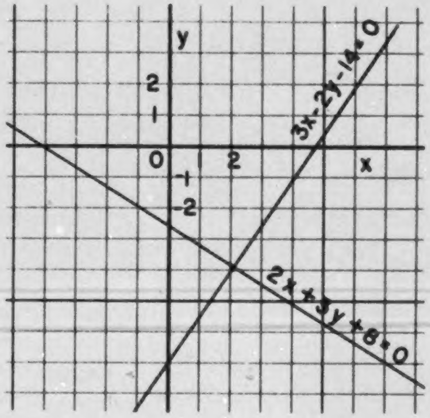
Page 473. Examples 1 and 2 illustrate both methods of obtaining the second simple sentence after we have the first one. The method of Example 1 is usually simpler, but sometimes the method of Example 2 is preferred where fractions are involved.

Help your students to see clearly that the sentence "$x = 3$ and $y = 2$" _is_ equivalent to "$4x - 3y = 16$ and "$2x + 5y = 16$".

The verification is not necessary logically to prove the sentences are equivalent. It is, however, desirable both to check accuracy of arithmetic and to keep before us what we mean by "the truth set of the sentence".

<ins>Answers to Problem Set 15-1b</ins>; pages 474-477:

1. (a) $\begin{cases} 3x - 2y - 14 = 0 \\ 2x + 3y + 8 = 0 \end{cases}$



$3(3x - 2y - 14) + 2(2x + 3y + 8) = 0$

$9x - 6y - 42 + 4x + 6y + 16 = 0$

$13x - 26 = 0$

$13x = 26$

$x = 2$

When $x = 2$,

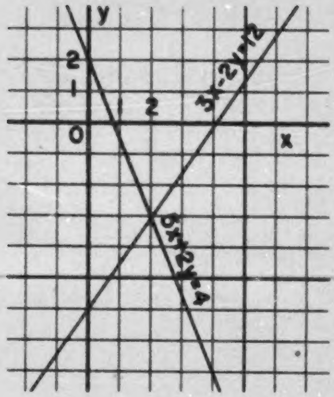
$2 \cdot 2 + 3y + 8 = 0$

$3y = -12$

$y = -4$

The truth set is $\{(2,-4)\}$.

Verification:	Left	Right
First Clause:	$3 \cdot 2 - 2(-4) - 14 = 6 + 8 - 14 = 0$	0
Second Clause:	$2 \cdot 2 + 3(-4) + 8 = 4 - 12 + 8 = 0$	0



(b) $\{(2,-3)\}$.

(c) $\{(5,-7)\}$

(d) $\{(17,12)\}$

(e) $\{(\frac{23}{2},\frac{37}{2})\}$

(f) $\{(-\frac{2}{3},\frac{1}{3})\}$

(g) $\{(\frac{3}{5}, -\frac{9}{5})\}$

(h) $\{(\frac{6}{5},\frac{7}{5})\}$

(i) $\{(15,16)\}$

2. At this point we are using the symbols c, d as specific numbers, namely the numbers which are assumed to make both sentences of the system true. Thus, we are not thinking of c and d as the usual variables, which represent unspecified numbers. The difference between "$3x - 2y - 5 = 0$" and "$3c - 2d - 5 = 0$" is that the first is an open sentence, whereas the second is a statement about two specific numerals naming the same number. The equations above simply constitute the arithmetic of finding common names of c and d, if they exist.

(a) Assume that (c,d) is a solution of the system

$$\begin{cases} x - 4y - 15 = 0 \\ 3x + 5y - 11 = 0 \end{cases}$$

Then

$$c - 4d - 15 = 0$$
$$3c + 5d - 11 = 0$$

$$3(c - 4d - 15) = 3\cdot 0$$
$$-1(3c + 5d - 11) = -1\cdot 0$$

$$3c - 12d - 45 = 0$$
$$-3c - 5d + 11 = 0$$

$$-17d - 34 = 0$$
$$d = -2$$

$$5(c - 4d - 15) = 5\cdot 0$$
$$4(3c + 5d - 11) = 4\cdot 0$$

$$5c - 20d - 75 = 0$$
$$12c + 20d - 44 = 0$$

$$17c - 119 = 0$$
$$c = 7$$

Check: $7 - 4(-2) - 15 = 7 + 8 - 15 = 0$

$3\cdot 7 + 5(-2) - 11 = 21 - 10 - 11 = 0$

Truth set: $\{(7,-2)\}$.

(b) $\{(\frac{1}{2}, 1)\}$

(c) $\{(\frac{7}{10}, \frac{4}{5})\}$

(d) Assume (c,d) is a solution to the system

$$\begin{cases} 2x + 2y = 3 \\ 3x + 3y = 4 \end{cases}$$

then

$$2c + 2d - 3 = 0$$
$$3c + 3d - 4 = 0$$

$$3(2c + 2d - 3) = 3\cdot 0$$
$$2(3c + 3d - 4) = 2\cdot 0$$

$$6c + 6d - 9 = 0$$
$$6c + 6d - 8 = 0$$

$0 + 0 - 1 = 0$. Since $-1 \neq 0$, this contradiction tells us that our assumption that (c,d) is a solution is false. Therefore, there is no solution. The student can further verify this conclusion by observing that these two lines have the same slope, but different intercepts: they are parallel and do not intersect.

3. (a) Suppose there were x tickets sold at 25 cents, and y tickets sold at 75 cents.
Open sentence: $x + y = 311$ and $25x + 75y = 10875$.
Truth set: $\{(249,62)\}$ Hence, there were 249 pupils and 62 adult tickets sold.

(b) If there are x girls, then Elsie has $x - 1$ sisters. Hence, there must be $x - 1$ boys. Similarly if there are y boys, then Jimmie has $y - 1$ brothers and $2(y - 1)$ sisters.
Open sentence: $x - 1 = y$ and $2(y - 1) = x$.
Truth set: $\{(4,3)\}$. Hence, there are 4 girls and 3 boys in the family.

(c) Suppose there were x three cent stamps purchased, and y four cent stamps.
Open sentence: $x + y = 352$ and $3x + 4y = 1267$.
Truth set: $\{(141,211)\}$. Hence, there were 141 three cent and 211 four cent stamps purchased.

(d) Suppose there were x one dollar bills and y five dollar bills.
Open sentence: $x + y = 154$ and $x + 5y = 465$.
Truth set: $\{(76\frac{1}{4},77\frac{3}{4})\}$. He has not counted correctly, since, the number of bills must be an integer.

4. The students should try to find the truth sets here by the method given on Pages 468-474. They will be successful with Problem 4(a) only. The method fails for Problems 4(b) and 4(c).

(b) $a(2x - y - 5) + b(4x - 2y - 10) = 0$
$(a + 2b)(2x) - (a + 2b)\, y - (a + 2b)5 = 0$

(c) $a(2x + y - 4) + b(2x + y - 2) = 0$
$(a + b)2x + (a + b)y - 4a - 2b = 0$

Clearly, in both cases, any choice of a and b which gives a zero coefficient for x does the same for y; and conversely.

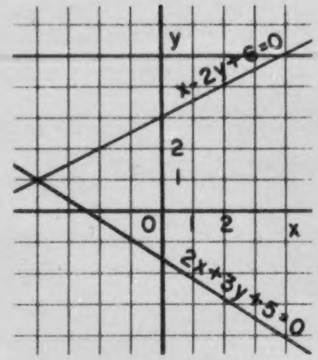
We ask the students to draw the graphs of the clauses of the systems so that they will make the following observations:

(1) In Problem 4(b), there is more than one point common to the graphs of the clauses $2x - y - 5 = 0$ and $4x - 2y - 10 = 0$;

(2) The graphs of the clauses in Problem 4(c) have no points in common.

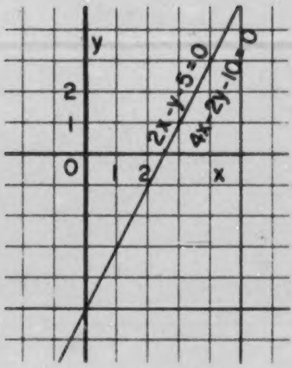
We should perhaps emphasize once again that the method on Pages 468-474 of the text leads to the truth set of the system if and only if the graphs of the clauses of the system have exactly one point in common.

4(a)



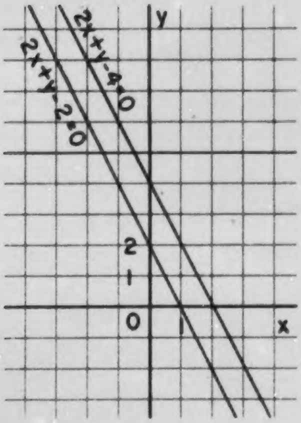
Truth set: {(-4,1)}

4(b)



Truth set: The whole line.

4(c)



Truth set is $\emptyset$:
The lines are parallel.

[page 477]

5. If the line $Ax + By + C = 0$ is to contain the origin then C must be equal to zero. Since $a(5x - 7y - 3) + b(3x - 6y + 5) = 0$ represents any line passing thru the intersection of the lines $5x - 7y - 3 = 0$, and $3x - 6y + 5 = 0$, we must choose a and b, not both 0, so that $-3a + 5b = 0$. One obvious choice is $a = 5$, and $b = 3$. Therefore, our compound sentence now becomes

$$5(5x - 7y - 3) + 3(3x - 6y + 5) = 0.$$

Therefore, $34x - 53y = 0$ is an equation of the line passing through the origin and passing through the intersection of the lines $5x - 7y - 3 = 0$ and $3x - 6y + 5 = 0$.

Page 477. For the system

$$\begin{cases} 2x - y - 5 = 0 \\ 4x - 2y - 10 = 0, \end{cases}$$

we have

$$a(2x - y - 5) + b(4x - 2y - 10) = (a + 2b)(2x) - (a + 2b)y - (a + 2b)5$$

It is then evident that no choice of a and b will give a 0 coefficient for one of x and y and not the other.

Page 478. The students' answers to the questions concerning the slope and y-intercept number of $y = -2x + 4$ and $y = -2x + 2$ should suggest why the y-form of an equation of a line is also called the slope-y-intercept form.

Page 478. When the algebraic technique we have developed is applied to the system

$$\begin{cases} 2x + y - 4 = 0 \\ 2x + y - 2 = 0, \end{cases}$$

we find that

$$a(2x + y - 4) + b(2x + y - 2) = (a + b)(2x) + (a + b)y - 4a - 2b$$

Clearly, any choice of a and b which gives a 0 coefficient for x also gives a 0 coefficient for y; and, conversely.

Page 480. The relation "are proportional to" for ordered sets of real numbers is a highly useful one. We have given the definition in the text for ordered pairs of real numbers. For ordered triples of numbers, it would read: The real numbers A, B, and C are proportional to the real numbers D, E, and F if there is a non-zero real number k such that

$$A = kD, \quad B = kE, \quad \text{and} \quad C = kF.$$

Thus, the lines of the system

$$\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0 \end{cases}$$

are coincident if and only if the numbers A, B, and C are proportional to D, E, and F.

The statement that A, B, and C are proportional to D, E, and F is often abbreviated to

$$\frac{A}{D} = \frac{B}{E} = \frac{C}{F}$$

where it is agreed that when one of the denominators is 0, then so is the corresponding numerator. It is by no means intended that one interpret $\frac{0}{0}$ as "the quotient of 0 by 0"; rather, $\frac{0}{0}$ is written for the statement $0 = k \cdot 0$ so that one can abbreviate such statements as "0, 1, and -7 are proportional to 0, -3, and 21" to

$$\frac{0}{0} = \frac{1}{-3} = \frac{-7}{21}.$$

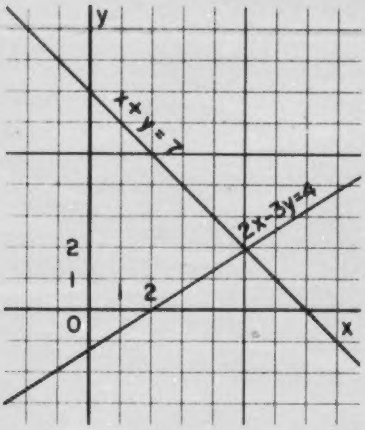
Using this abbreviation, the lines of the system

$$\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0 \end{cases} \quad \text{are coincident if and only if} \quad \frac{A}{D} = \frac{B}{E} = \frac{C}{F}.$$

Answers to Problem Set 15-1c; pages 480-484:

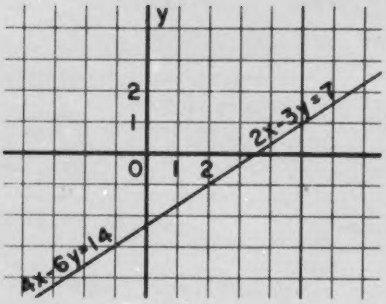
1.

Example 1.



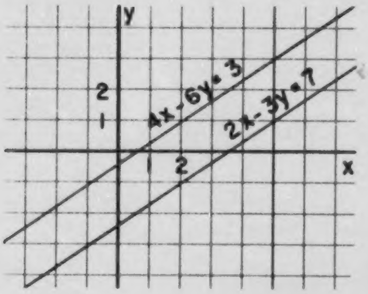
Truth set: $\{(5,2)\}$

Example 2.



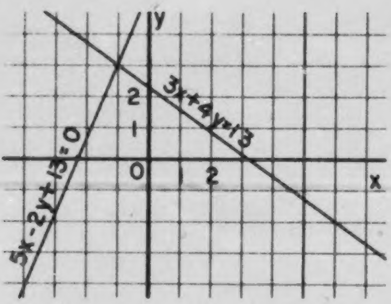
Truth set is whole line.

Example 3.



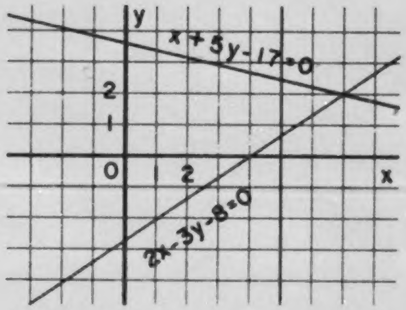
Truth set is $\emptyset$:
Lines are parallel.

2(a)



Truth set: $\{(-1,4)\}$

2(b)



Truth set: $\{(7,2)\}$

(c) The set of all coordinates of points on the line.

(d) $\{(\frac{1}{2}, \frac{1}{4})\}$

(e) $\emptyset$

(f) $\{(\frac{28}{5}, 3)\}$

3. The idea here is that the values of y for each equation must be the same when x is the abscissa of the point common to the lines of the system

$$\begin{cases} 2x - y - 7 = 0 \\ 5x + 2y - 4 = 0 \end{cases}$$

This simple observation then allows one to solve an equation in the one variable x, obtained by writing each equation in its y-form and then "equating" these two expressions in x.

Instead of writing both equations in their y-forms, we sometimes find it more convenient to write one in its y-form and replace y by the resulting expression in x in the other equation.

Encourage your students to use whichever one of these methods seems more appropriate.

(a) $\begin{cases} 3x + y + 18 = 0 \\ 2x - 7y - 34 = 0 \end{cases}$

From the first equation

$$y = -3x - 18$$

Then

$$\begin{aligned} 2x - 7(-3x - 18) - 34 &= 0 \\ 2x + 21x + 126 - 34 &= 0 \\ 23x + 92 &= 0 \\ x &= -4 \end{aligned}$$

When $x = -4$,

$$y = -3(-4) - 18$$

$$y = -6$$

The solution is $(-4,-6)$.

(b) $\begin{cases} y = \frac{2}{3}x + 2 \\ y = -\frac{5}{2}x + 40 \end{cases}$

$$\frac{2}{3}x + 2 = -\frac{5}{2}x + 40$$

$$4x + 12 = -15x + 240$$

$$19x = 228$$

$$x = 12$$

When $x = 12$,

$$y = \frac{2}{3}\cdot 12 + 2$$

$$y = 10$$

The solution is $(12,10)$.

(c) $\begin{cases} 5x + 2y - 4 = 0 \\ 10x + 4y - 8 = 0 \end{cases}$

Since $\frac{5}{10} = \frac{2}{4} = \frac{-4}{-8}$,

The equations represent the same line, and the truth set is the set of the coordinates of all points on the line.

(d) $(3,-5)$

(e) $\begin{cases} x + 7y = 11 \\ x - 3y = -4 \end{cases}$

$\begin{cases} x = -7y + 11 \\ x = 3y - 4 \end{cases}$

$$-7y + 11 = 3y - 4$$
$$15 = 10y$$
$$\frac{3}{2} = y$$

When $y = \frac{3}{2}$,

$$x = 3(\tfrac{3}{2}) - 4$$
$$x = \frac{1}{2}$$

The solution is $(\frac{1}{2}, \frac{3}{2})$.

(f) $(-2,\frac{4}{3})$

4. (a) The graphs of the clauses intersect in the point which corresponds to the given ordered pair.

(b) The graphs coincide. The two clauses are equivalent. There is a non-zero real number k such that $Ax + By + C = k(Dx + Ey + F)$.

(c) A and B are proportional to D and E. The graphs are parallel.

[pages 482-483]

5. We find the truth set to be $\{(\frac{3}{2},\frac{5}{2})\}$ by any of the methods:

   (a) Choose a,b such that

   $a(4x + 2y - 11) + b(3x - y - 2) = 0$

   is equation of line through intersection and parallel to an axis, etc.

   (b) Assume (c,d) satisfies both equations and solve resulting equations for c,d by addition method.

   (c) Write both equations in y-form and equate values of y.

   (d) Use the substitution method.

   (e) Draw graphs of equations and estimate coordinates of point of intersection.

6. (a) $\{(3,-4)\}$. Any method, except y-forms, would be equally effective.

   (b) $\{(0,0)\}$. Addition method would be easiest.

   (c) $\{(\frac{1}{2},\frac{1}{3})\}$. Substitution method; or clear fractions and use addition method.

   (d) $\{(1, \frac{1}{2})\}$. Addition method because the coefficients of y are opposites.

   (e) $\{(6,1)\}$. Solve for x in first equation; substitute this value in second.

   (f) $\emptyset$. Simplify left members and notice that the coefficients of x and y are proportional.

   (g) $\{(\frac{7}{3},\frac{8}{9})\}$. Any method except graphing.

   (h) $\{(5,7)\}$. Substitution method because the first equation can be easily "solved for y in terms of x".

In the following problems try to bring out the possibility of using either two variables or one variable. Do some of the problems both ways. Look at the relative metits of the two ways. See how the equation in one variable develops from the two equations in two variables.

7. If the numbers are x and y, then

$$\begin{cases} x + y = 56 \\ x - y = 18 \end{cases}$$

Truth set: $\{(37,19)\}$

The numbers are 37 and 19.

OR If the larger number is x, the smaller number is $56 - x$, and

$$x - (56 - x) = 18 .$$

OR If the smaller number is y, the larger number is $y + 18$, and

$$y + (y + 18) = 56 .$$

8. If Sally is x years old and Joe is y years old, then

$$\begin{cases} x + y = 30 \\ x - y = 4 \end{cases} \quad \text{OR} \quad \begin{cases} x + y = 30 \\ y - x = 4 \end{cases}$$

Notice that we do not know which is the older, so the problem can be answered in two ways. Notice also that the information "In five years" is irrelevant since the difference in their ages is the same now as it is at any time.

Truth set: $\{(17,13)\}$ or $\{(13,17)\}$

Sally is 17 years old and Joe is 13 or

Joe is 17 years old and Sally is 13.

This problem can be set up in four different ways using one variable:

$$\begin{cases} \text{Sally} \quad n \quad \text{years old} \\ \text{Joe} \quad 30-n \quad \text{years old} \end{cases} \qquad \begin{cases} \text{Sally} \quad n \quad \text{years old} \\ \text{Joe} \quad (n+4) \text{ or } (n-4) \text{ years old} \end{cases}$$

$$\begin{cases} \text{Joe} \quad n \quad \text{years old} \\ \text{Sally} \quad 30-n \quad \text{years old} \end{cases} \qquad \begin{cases} \text{Joe} \quad n \quad \text{years old} \\ \text{Sally} \quad (n+4) \text{ or } (n-4) \text{ years old} \end{cases}$$

9. If he uses a pounds of almonds and c pounds of cashews, then

$$\begin{cases} a + c = 200 \\ 1.50a + 1.20c = 1.32(200) \end{cases}$$

Truth set: $\{(80,120)\}$

He should use 80 pounds of almonds and 120 pounds of cashews.

OR If he uses a pounds of almonds, then he uses (200 - a) pounds of cashews and

$$1.50a + 1.20(200 - a) = 1.32(200)$$

10. If the tens' digit is t and the units' digit is u, then

$$\begin{cases} u = 2t + 1 \\ (10t + u) + u = 3t + 35 \end{cases}$$

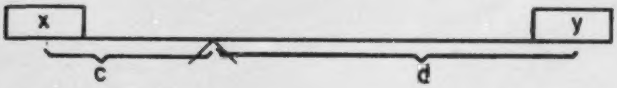
Truth set: $\{(3,7)\}$

The number is 37.

OR If u is the units' digit, then (2u + 1) is the tens' digit, and

$$10(2u + 1) + a = 3(2u + 1) + 35$$

11. We use the principle from Physics that a lever



balances if $cx = dy$

where c and d measure the distances from the fulcrum, or point of balance, and x and y measure the weights of the objects on the lever. We can think of cx and dy as measures of the forces which tend to turn the lever.

Explain this informally to the students but do not make a major lesson in physics of it.

If Hugh is h feet from the point of balance and Fred is f feet from the point of balance, then

$$\begin{cases} f + h = 9 \\ 100f = 80h \end{cases}$$

The truth set: $\{(4,5)\}$

Hugh is 5 feet, Fred is 4 feet from the point of balance.

12. If one boy weighs a pounds and the other boy weighs b pounds, then

$$\begin{cases} a + b = 209 \\ 5a = 6b \end{cases}$$

Truth set: $\{(114,95)\}$

One boy weighs 114 pounds, the other 95 pounds.

13. If the speed of the current is c m.p.h. and the speed of the boat in still water is b m.p.h., then

$$\begin{cases} \frac{3}{2}(b + c) = 12 \\ 6(b - c) = 12 \end{cases}$$

Truth set: $\{(5,3)\}$

The speed of the current is 3 m.p.h. and the speed of the boat in still water is 5 m.p.h.

This problem is not easily done with one variable.

14. If apples cost a cents per pound, and bananas cost b cents per pound, then

$$\begin{cases} 3a + 4b = 108 \\ 4a + 3b = 102 \end{cases}$$

Truth set: $\{(12,18)\}$

Apples are 12 cents per pound and bananas are 18 cents per pound.

15. If A walks at a miles per hour, and
B walks at b miles per hour, then
in 60 hours, A walks 60a miles and
B walks 60b miles;
in 5 hours, A walks 5a miles and
B walks 5b miles.

$$\begin{cases} 60a = 60b + 30 \\ 5a + 5b = 30 \end{cases}$$

The truth set: $\{(\frac{13}{4}, \frac{11}{4})\}$

A walks at $3\frac{1}{4}$ miles per hour.

B walks at $2\frac{3}{4}$ miles per hour.

16. If there are x quarts of the 90% solution and y quarts of the 75% solution, then there are .90x quarts of alcohol in the 90% solution and .75y quarts of alcohol in the 75% solution. Furthermore, there are .78(20) quarts of alcohol in the mixed solution.

$$\begin{cases} .90x + .75y = .78(20) \\ x + y = 20 \end{cases}$$

Truth set: $\{(4,16)\}$
He should use 4 quarts of the 90% solution.

17. If the average speed of car A is a miles per hour and the average speed of car B is b miles per hour, then

$$\begin{cases} \frac{300}{a} = \frac{275}{b} - \frac{1}{2} \\ \frac{300}{a} = \frac{240}{b} + \frac{1}{5} \end{cases}$$

Truth set: $\{(60,50)\}$

A's average speed was 60 m.p.h. and
B's average speed was 50 m.p.h.

Page 485. The system of inequalities denoted by

$$\begin{cases} x + 2y - 4 > 0 \\ 2x - y - 3 > 0 \end{cases}$$

is a shorthand for the compound open sentence

$$x + 2y - 4 > 0 \text{ and } 2x - y - 3 > 0.$$

The graph of this system is, of course, the graph of the compound open sentence.

Page 485. In drawing the graph of an inequality

$$Ax + By + C > 0, \quad (B \neq 0)$$

write the inequality in the form

$$(1) \quad y > -\frac{A}{B}x - \frac{C}{B} \quad (B > 0),$$

$$(2) \quad y < -\frac{A}{B}x - \frac{C}{B} \quad (B < 0),$$

whichever is appropriate. On the line

$$(3) \quad Ax + By + C = 0,$$

$y = -\frac{A}{B}x - \frac{C}{B}$. Thus, for a given value of x, (x,y) satisfies equation (1) if the point (x,y) is above the line, since the ordinate of this point is greater than $-\frac{A}{B}x - \frac{C}{B}$. On the other hand, for a given value of x, (x,y) satisfies equation (2) if the point (x,y) is below the line. In general, the graph of (1) is the set of points above the line (3); the graph of (2) is the set of points below the line (3).

Page 486. The graph of the system

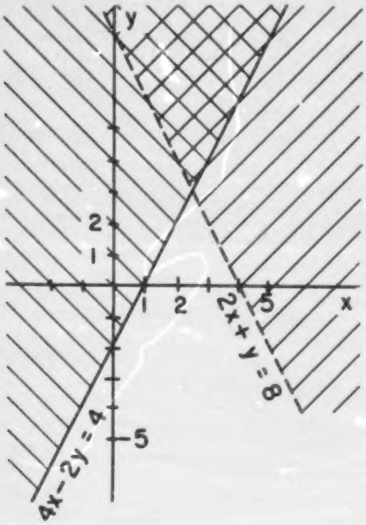
$$\begin{cases} 3x - 2y - 5 = 0 \\ x + 3y - 9 \leq 0 \end{cases}$$

is the portion of the line $3x - 2y - 5 = 0$ which is below or on the line $x + 3y - 9 = 0$.

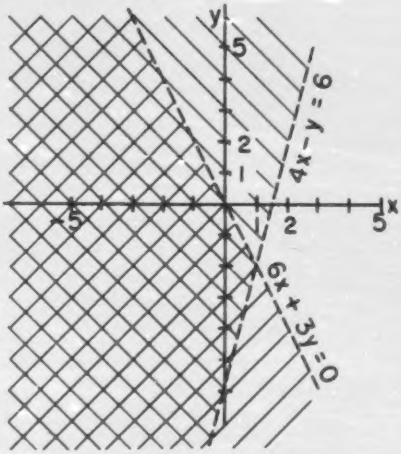
Answers to Problem Set 15-2a; page 487:

The truth set of each open sentence in Problems 1, 2, 5, 6, 7 consists of all points in the doubly shaded regions of the graph, together with all points of the solid line boundaries of these regions. In Problems 3 and 4, the truth set is the portion of the solid line inside the shaded region.

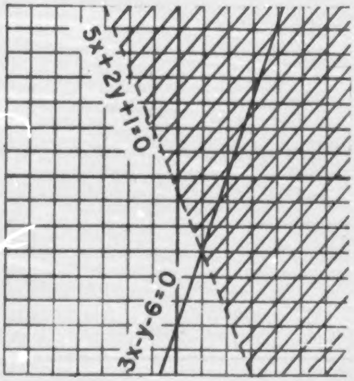
1.



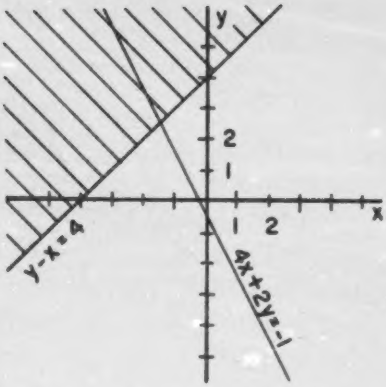
2.



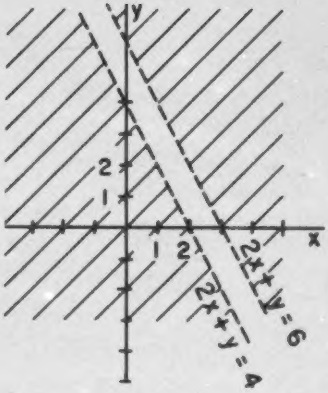
3.



4.

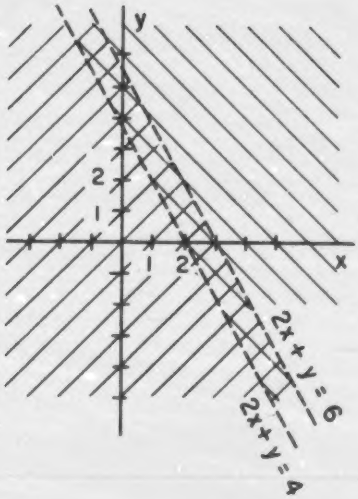


5.

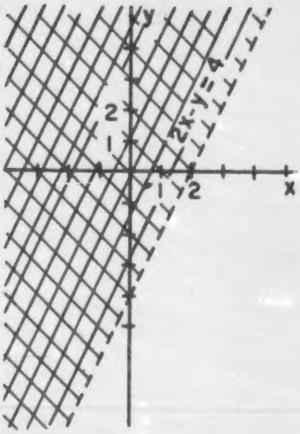


The empty set, ∅.

6.



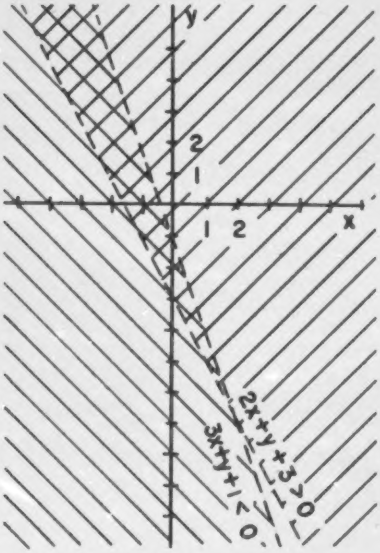
7.



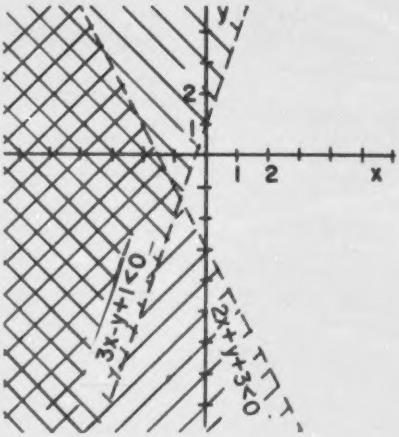
Answers to Problem Set 15-2b; page 488:

The graphs of all the open sentences in Problems 1-3 of this set of problems consists of all points in all the shaded areas together with all points of the solid line boundaries of these regions. In Problem 4, the graph is the doubly shaded region.

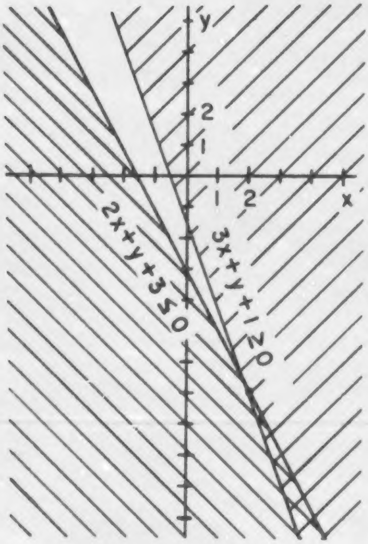
1.



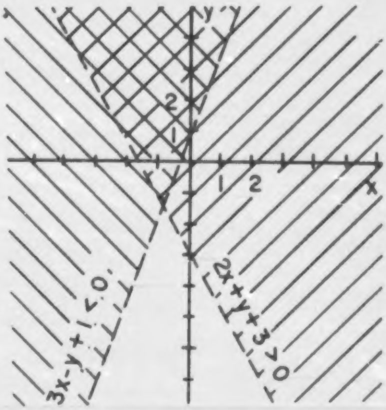
2.



3.



4.



Page 488. The graph of

$$(x - y - 2)(x + y - 2) < 0$$

is the graph of

$$x - y - 2 < 0 \quad \text{and} \quad x + y - 2 > 0$$

or

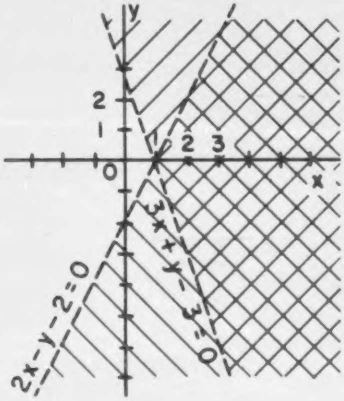
$$x - y - 2 > 0 \quad \text{and} \quad x + y - 2 < 0.$$

The graph consists of the two regions which are singly shaded.

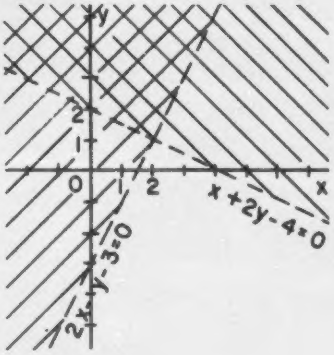
Answers to Problem Set 12-2c; pages 489-490;

The graphs of the truth sets here consist of all points in the unshaded and the doubly shaded regions of the figures.

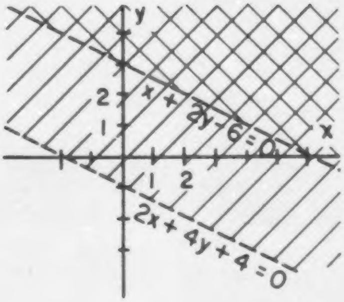
1(a)


y
2
1
0
1
2
3
x
3x+y-3=0
2x-y-2=0

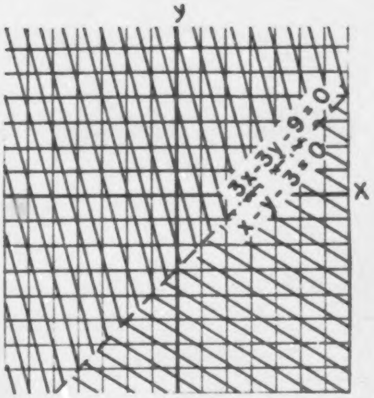
1(b)


y
2
1
0
1
2
x
x+2y-4=0
2x-y-3=0

1(c)

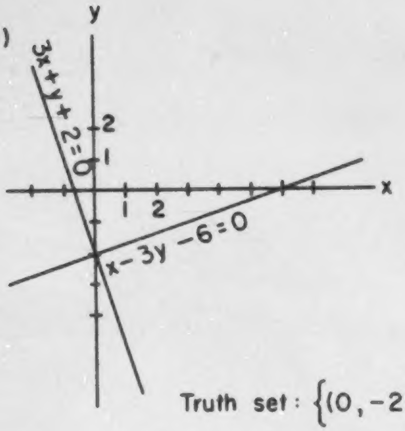

y
2
1
x+2y-6=0
x
1
2
2x+4y+4=0

1(d)


y
x
3x-3y-9=0
x-y-3=0

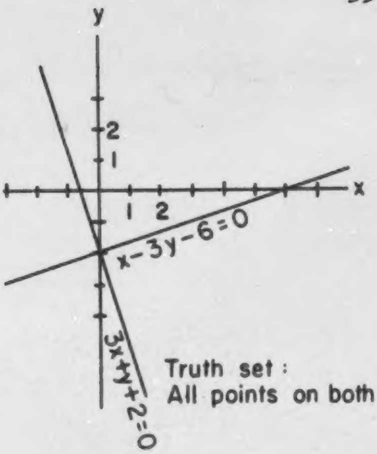
The null set, $\emptyset$.

2(a)


y
x
2
1
1 2
3x+y+2=0
x-3y-6=0

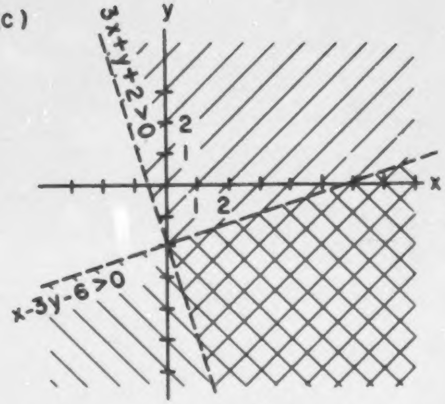
Truth set: $\{(0,-2)\}$

2(b)


y
x
2
1
1 2
x-3y-6=0
3x+y+2=0

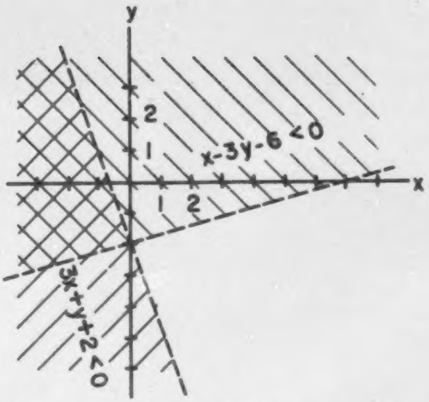
Truth set:
All points on both lines.

2(c)


y
x
2
1
1 2
3x+y+2>0
x-3y-6>0

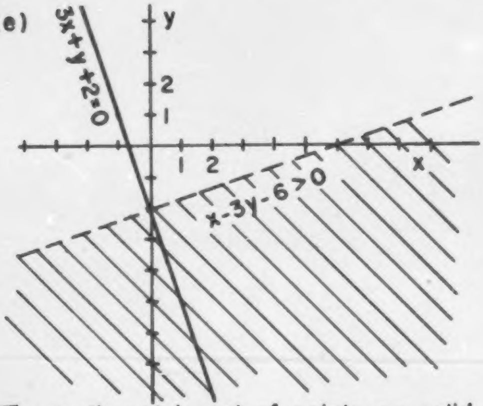
The truth set is doubly shaded region.

2(d)


y
x
2
1
1 2
x-3y-6<0
3x+y+2<0

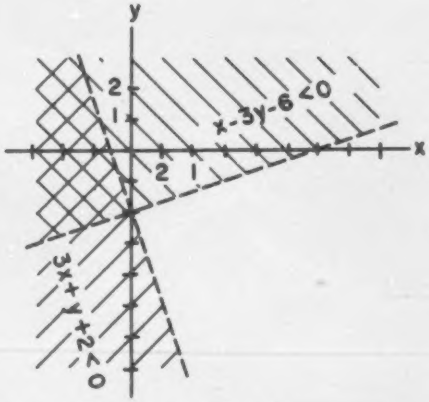
The truth set is doubly shaded region

2(e)


y
x
2
1
1 2
3x+y+2=0
x-3y-6>0

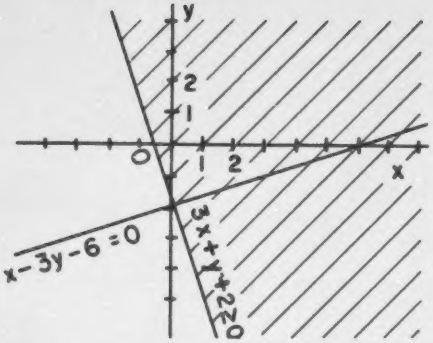
The truth set is set of points on solid line within shaded region

2 (f)


y
x
2
1
2 1
x-3y-6<0
3x+y+2<0

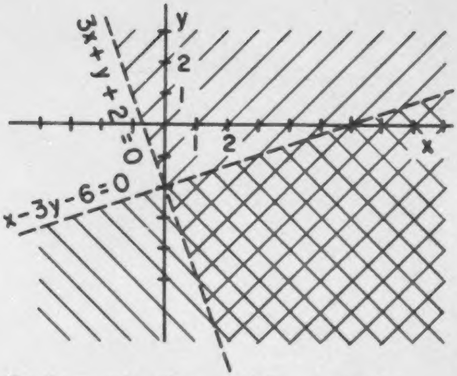
The truth set is whole shaded region.

2(g)



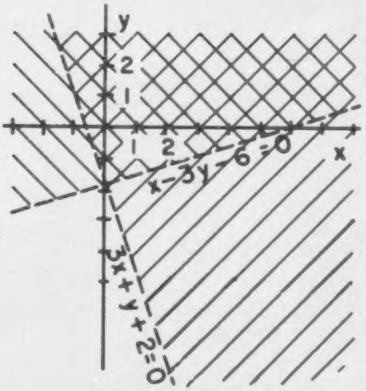
Truth set is whole shaded area and both lines

2(h)



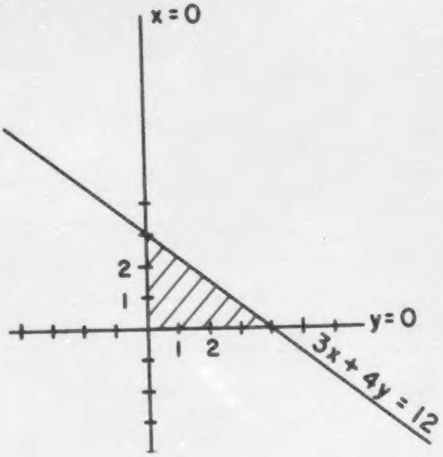
Truth set is the doubly shaded and the unshaded region.

2(i)

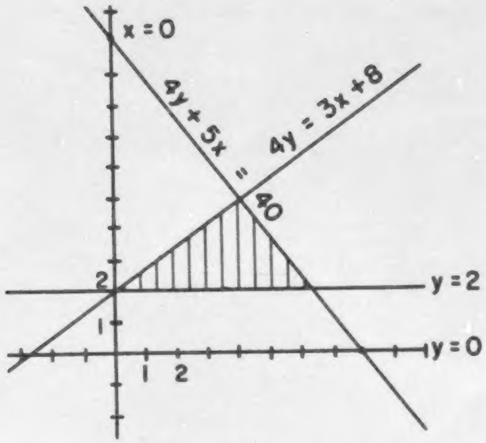


Truth set is the doubly shaded and the unshaded regions

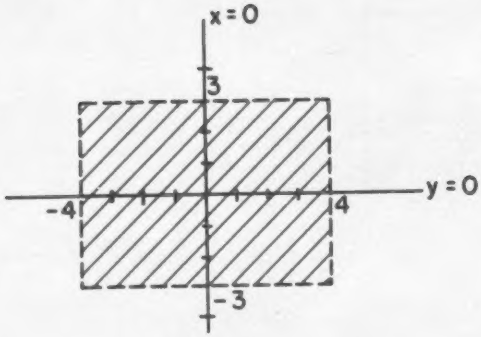
3(a)


x = 0
2
1
1
2
y = 0
3x + 4y = 12

3(b)


x = 0
4y + 5x = 40
4y = 3x + 8
2
1
y = 2
1
2
y = 0

3(c)



*4. If r is the number of running plays and p is the number of pass plays, then $3r$ is the number of yards made on r running plays and $20(\frac{1}{3})$ p is the number of yards made on p passing plays. Since, the team is 60 yards from the goal line,

$$3r + \frac{20}{3}p \geq 60$$

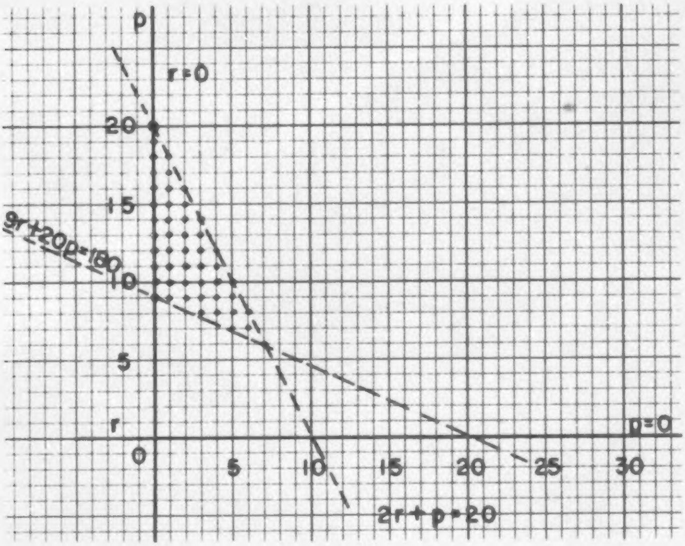
if they are to score.
$30r$ seconds are required for r running plays, and $15p$ seconds are required for p passing plays; therefore,

$$30r + 15p \leq 5(60)$$

These two inequalities give the equivalent system

$$\begin{cases} 20p + 9r \geq 180 \\ p + 2r \leq 20 \end{cases}$$ (p and r are non-negative integers.)

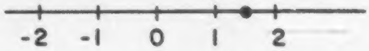
The graph of this system is



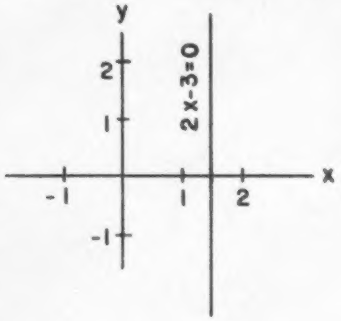
It is evident there are 48 different combinations of r and p which will assure success, for example, 2 running and 10 passing, etc. However, there are some combinations which leave a smaller time remaining, thus giving the opponents less time to try to score. These are the points of the graph nearest the line $p + 2r = 20$.

Answers to Review Problems; pages 490-492:

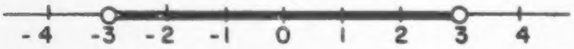
1. (a) As an equation in one variable, the truth set of "$2x - 3 = 0$" is $\{(\frac{3}{2})\}$ and its graph is:



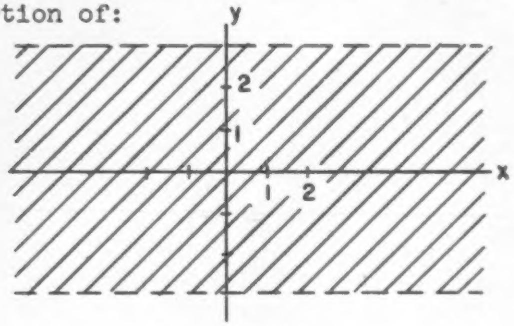
(b) As an equation in two variables, its truth set is the set of all ordered pairs of real numbers with first number $\frac{3}{2}$. Its graph is



2. (a) As a sentence in one variable, the truth set of "$|y| < 3$" is the set of all y such that $-3 < y < 3$, and its graph is:



(b) As a sentence in two variables, its truth set is the set of all ordered pairs of real numbers with second number between -3 and 3. The graph is the shaded portion of:



3. (a) The sentences are *not* equivalent, because the truth set of "$x - 2 = 0$ *and* $x - 1 = 0$" is $\emptyset$ (x cannot be both 2 and 1.)

   (b) They are equivalent. The operation of multiplying by $x^2 + 1$ leads to an equivalent sentence.

   (c) Not equivalent. The graph of $x\,y > 0$ contains all points in the first and third quadrants, whereas the graph of "$x > 0$ or $y > 0$" contains all points in quadrants I, II, and IV.

   (d) Not equivalent. The truth set of $(y - 1) = 2(x-1)$ includes the element $(1,1)$, whereas the truth set of $\frac{y-1}{x-1} = 2$ does not include $(1,1)$.

   (e) They are equivalent, Both have the truth set $\{(6,-3)\}$.

4. The line $a(3x - 5y - 4) + b(2x + 3y + 4) = 0$ contains the point of intersection for any numbers a and b, not both 0. Let $a = -2$ and $b = 3$. Then

$$-2(3x - 5y - 4) + 3(2x + 3y + 4) = 0$$
$$19y + 20 = 0$$

is the equation of the required horizontal line. Let $a = 3$ and $b = 5$. Then,

$$3(3x - 5y - 4) + 5(2x + 3y + 4) = 0$$
$$19x + 8 = 0$$

is the equation of the required vertical line.

5. (a) $\{(5,3)\}$

   (b) $\{(-2,-1)\}$

   (c) Set of all solutions of either equation.

   (d) $\emptyset$

   (e) $\{(6,-5)\}$

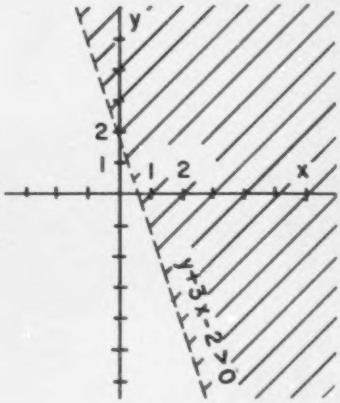
   (f) $\{(24,9)\}$

6. (a) The lines $Ax + By + C = 0$ and $Dx + Ey + F = 0$ are parallel if and only if either $B = E = 0$, $\frac{A}{D} \neq \frac{C}{F}$ or $\frac{A}{D} = \frac{B}{E} \neq \frac{C}{F}$.

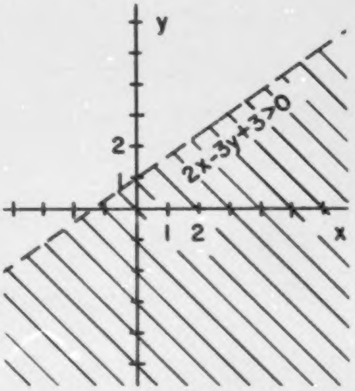
(b) If $\frac{A}{D} = \frac{B}{E} = \frac{C}{F}$ the equations represent the same non-vertical line.

(c) Two lines have exactly one common point if and only if both have slopes and their slopes are different, or if one has a slope and the other does not.

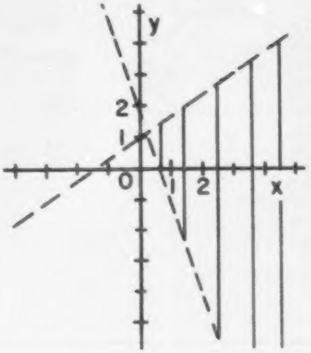
7. (a)


y
2
1
1
2
x
y+3x-2>0

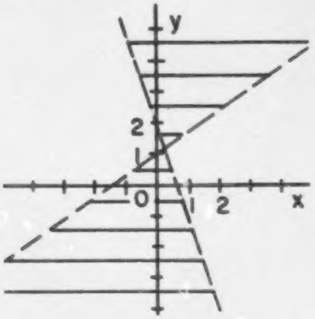
(b)


y
2
2x-3y+3>0
1
2
x

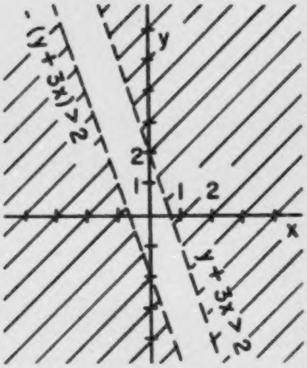
(c)



(d)



(e)



"$|y + 3x| > 2$" is equivalent to "$y + 3x > 2$ or $-(y + 3x) > 2$".

8. (a) $\begin{cases} y = x + 1 \\ x + y = 57, \quad x \text{ and } y \text{ integers.} \end{cases}$

Truth set: $\{(28,29)\}$

(b) $\begin{cases} x + y = 16 \\ 2x = y - 3, \quad x \text{ and } y \text{ integers.} \end{cases}$

Truth set: $\emptyset$

(c) $\begin{cases} x + y = 45 \\ y = 4x + 5 \end{cases}$ Truth set: $\{(8,37)\}$

(d) $\begin{cases} x + y = 20 \\ 4.8x + 6.0y = 110. \end{cases}$ Truth set $\{(8\frac{1}{3}, 11\frac{2}{3})\}$

Suggested Test Items

1. Draw the graphs of the truth sets of the following sentences:

   (a) $x = 2$ and $2x - 3y + 5 = 0$

   (b) $y - 3 = 0$ or $x + y = 1$

   (c) $(2x - 1)(x + y) = 0$

2. Consider the system of equations,

$$\begin{cases} 3x + y - 5 = 0 \\ 2x - 3y + 4 = 0. \end{cases}$$

   (a) Estimate the truth set of this system by drawing graphs of the two equations.

   (b) Select a value of a and a value of b such that

   $a(3x + y - 5) + b(2x - 3y + 4) = 0$

   is the equation of a horizontal line containing the point of intersection of the two given lines.

   (c) Write each of the given equations in y-form and solve the equation obtained by equating these two resulting expressions in x.

   (d) Explain the relationship between the results of parts (a), (b) and (c).

3. Solve the following systems by any methods. In each case explain why you chose a particular method.

   (a) $\begin{cases} y = -2x + 1 \\ y = 3x - 4 \end{cases}$

   (b) $\begin{cases} y = -2x + 1 \\ 3x + 2y = 4 \end{cases}$

   (c) $\begin{cases} x - 2y + 3 = 0 \\ 4x - 5y + 6 = 0 \end{cases}$

(d) $\begin{cases} \frac{1}{2}x - y + 3 = 0 \\ x - \frac{1}{3}y + 1 = 0 \end{cases}$

(e) $\begin{cases} \frac{1}{2}x - y + 3 = 0 \\ x - 2y - 3 = 0 \end{cases}$

(f) $\begin{cases} y - 3x = 3 \\ x - \frac{1}{3}y + 1 = 0 \end{cases}$

4. Without solving, determine which of the following systems have exactly one solution, which have many solutions, which have no solution.

(a) $\begin{cases} 3x = 4y - 2 \\ 8y + 6x + 2 = 0 \end{cases}$

(b) $\begin{cases} \frac{1}{2}x + 3 = 2y \\ x - 2y = 2 \end{cases}$

(c) $\begin{cases} 1\frac{1}{2} + 2\frac{1}{2}y = x \\ 3 = x + 2.5y \end{cases}$

(d) $\begin{cases} \frac{3}{2}x - \frac{4}{3}y - 1 = 0 \\ \frac{2}{3}x - \frac{3}{4}y + 2 = 0 \end{cases}$

(e) $\begin{cases} .5y = .4 - .4x \\ 8x + 10y = 8 \end{cases}$

(f) $\begin{cases} \frac{3}{5}x + \frac{1}{2}y = 1 \\ .12x + y = 2 \end{cases}$

In problems 5-10, translate into open sentences and solve.

5. Two numbers are such that their difference is 3 and their average is 6. What are the numbers?

6. Can two integers be found whose difference is 13 and the sum of whose successors is 28?

7. The digits of an integer between 0 and 100 have the sum 12, and the tens digit is 3 more than twice the units digit. What is the integer?

8. If the digits of an integer between 0 and 100 are reversed, the resulting integer is 45 greater than the given integer. What is the integer if the sum of its digits is 11?

9. A man made two investments, the first at 4% and the second at 6%. He received a yearly income from them of \$400. If the total investment was \$8,000, how much did he invest at each rate?

10. How many pounds each of 95-cent and 90-cent coffee must be mixed to obtain a mixture of 90 pounds to be sold for 92 cents per pound?

11. With respect to separate sets of axes draw the graphs of:

(a) $\begin{cases} 2x + 3 > 0 \\ y - 3 < 0 \end{cases}$

(b) $\begin{cases} y < x + 1 \\ 2x - y < 0 \end{cases}$

(c) $2x + 3 > 0$ or
$y - 3 < 0$

(d) $y < x + 1$ or
$2x - y < 0$

(e) $(x + y - 1)(2x - y) > 0.$

Chapter 16

QUADRATIC POLYNOMIALS

This chapter continues the work on graphs which was started in Chapter 14. After graphing some general quadratic polynomials of the form $Ax^2 + Bx + C$ by drawing curves through selected points, we examine the parabola which is the graph of the simple polynomial x^2. We note the changes in the graph as x^2 is multiplied by a, as some number h is added to x, and as some number k is added to x^2. We then generalize about the graph of $a(x-h)^2 + k$ where a, h, and k are real numbers and $a \neq 0$. This leads us to use the method of completing the square to change any quadratic polynomial into the standard form $a(x-h)^2 + k$. We finally use the method of completing the square to solve quadratic equations.

16-1 Graphs of Quadratic Polynomials

Page 493. $-2(x + 1)^2 + 3$ is a quadratic polynomial, since it is equal to $-2x^2 - 4x + 1$ for all values of x.

Example 1.

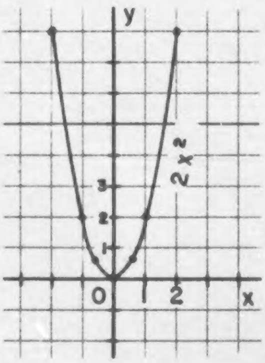
$$y = x^2 - 2x - 3$$

x	-2	$-\frac{3}{2}$	-1	$-\frac{2}{3}$	0	$\frac{1}{2}$	1	$\frac{4}{3}$	2	$\frac{5}{2}$	3	4
y	5	$\frac{9}{4}$	0	$-\frac{11}{9}$	-3	$-\frac{15}{4}$	-4	$-\frac{35}{9}$	-3	$-\frac{7}{4}$	0	5

Answers to Problem Set 16-1a; page 494:

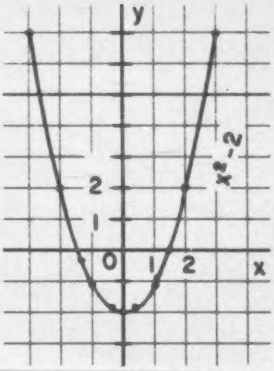
1.

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	I	2
y	8	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2	8



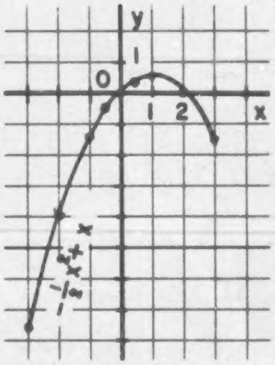
2.

x	-3	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{3}$	1	2	3
y	7	2	$\frac{1}{4}$	-1	$-\frac{7}{4}$	-2	$-\frac{17}{9}$	-1	2	7



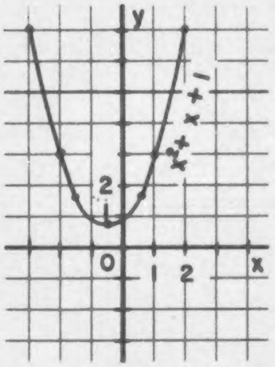
3.

x	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3
y	$-7\frac{1}{2}$	-4	$-\frac{3}{2}$	$-\frac{5}{8}$	0	$\frac{3}{8}$	$\frac{1}{2}$	0	$-\frac{3}{2}$



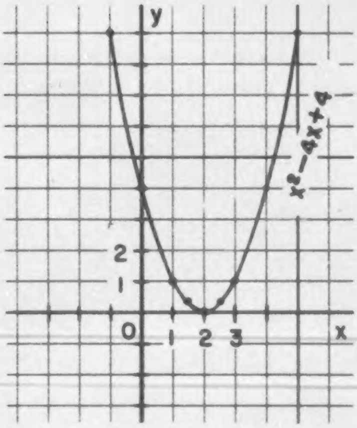
4.

x	-3	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	7	3	$\frac{7}{4}$	1	$\frac{3}{4}$	1	$\frac{7}{4}$	3	7



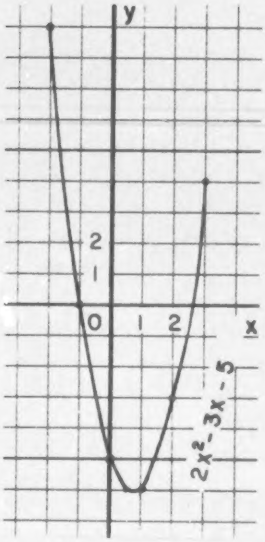
5.

x	-1	0	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	4	5
y	9	4	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	4	9



6.

x	-2	-1	0	$\frac{1}{2}$	$\frac{3}{4}$	1	2	3
y	9	0	-5	-6	$-6\frac{1}{8}$	-6	-3	4



Page 495.

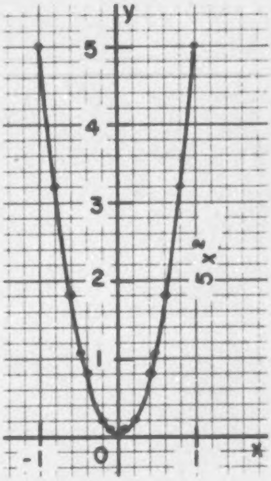
x	-3	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	-0.1	0	$\frac{1}{3}$	1	$\frac{4}{3}$	2	3
x^2	9	4	$\frac{9}{4}$	1	$\frac{1}{4}$	$.01$	0	$\frac{1}{9}$	1	$\frac{16}{9}$	4	9
$2x^2$	18	8	$\frac{9}{2}$	2	$\frac{1}{2}$	$.02$	0	$\frac{2}{9}$	2	$\frac{32}{9}$	8	18
$\frac{1}{2}x^2$	$\frac{9}{2}$	2	$\frac{9}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$.005$	0	$\frac{1}{18}$	$\frac{1}{2}$	$\frac{8}{9}$	2	$\frac{9}{2}$
$-\frac{1}{2}x^2$	$-\frac{9}{2}$	-2	$-\frac{9}{8}$	$-\frac{1}{2}$	$-\frac{1}{8}$	$-.005$	0	$-\frac{1}{18}$	$-\frac{1}{2}$	$-\frac{8}{9}$	-2	$-\frac{9}{2}$

Answers to Problem Set 16-1b; page 496:

1. The graph of $2x^2$ can be obtained by multiplying each ordinate of x^2 by 2.
2. The graph of $-\frac{1}{2}x^2$ can be obtained by rotating the graph of $\frac{1}{2}x^2$ one-half revolution about the x-axis.
3.

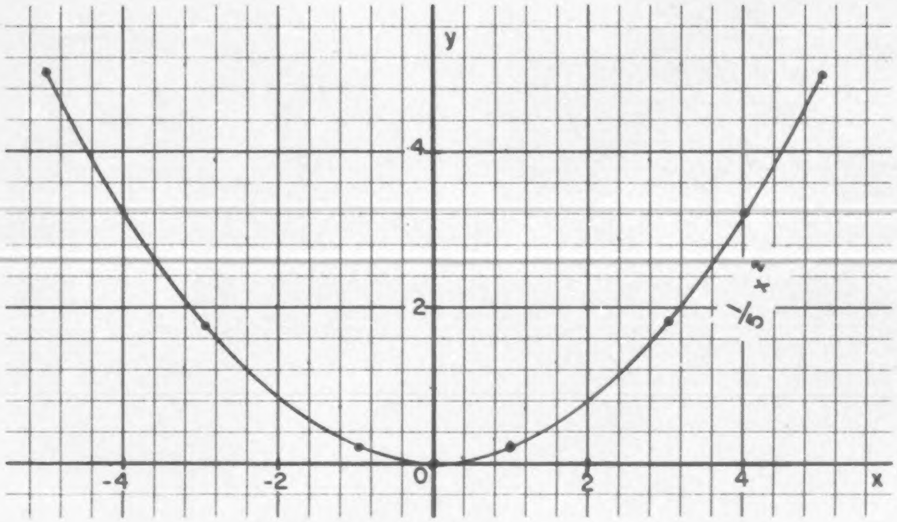
x	-1	$-.8$	$-.6$	$-.5$	$-.4$	$-.2$	$-.1$
y	5	3.2	1.8	1.25	$.8$	$.2$	$.05$

x	0	$.1$	$.2$	$.4$	$.5$	$.6$	$.8$	1
y	0	$.05$	$.2$	$.8$	1.25	1.8	3.2	5



4.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	5	3.2	1.8	.8	.2	0	.2	.8	1.8	3.2	5



5. The graph of $-5x^2$ can be obtained by rotating the graph of $5x^2$ one-half revolution about the x-axis.

6. The graph of $-ax^2$ can be obtained by rotating the graph of ax^2 one-half revolution about the x-axis.

Page 497.

$$y = \frac{1}{2}(x - 3)^2$$

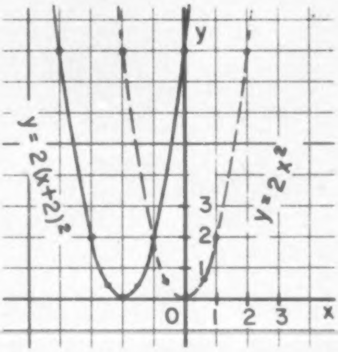
x	0	$\frac{1}{3}$	1	$\frac{3}{2}$	2	2.5	3	$\frac{13}{4}$	4	5
y	$\frac{9}{2}$	$\frac{32}{9}$	2	$\frac{9}{8}$	$\frac{1}{2}$	.125	0	$\frac{1}{32}$	$\frac{1}{2}$	2

We can obtain the graph of $y = -(x + 3)^2$ by moving the graph of $y = -x^2$ three units to the left.

Answers to Problem Set 16-1c; page 498:

1. $y = 2(x + 2)^2$

x	-4	-3	$-\frac{5}{2}$	-2	$-\frac{3}{2}$	-1	0
y	8	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2	8



You can obtain the graph of $y = 2(x + 2)^2$ by moving the graph of $y = 2x^2$ two units to the left.

2. (a) You can obtain the graph of $y = 3(x + 4)^2$ by moving the graph of $y = 3x^2$ four units to the left.
   (b) You can obtain the graph of $y = -2(x - 3)^2$ by moving the graph of $y = -2x^2$ three units to the right.
   (c) You can obtain the graph of $y = -\frac{1}{2}(x + 1)^2$ by moving the graph of $y = -\frac{1}{2}x^2$ one unit to the left.
   (d) You can obtain the graph of $y = \frac{1}{3}(x + \frac{1}{2})^2$ by moving the graph of $y = \frac{1}{3}x^2$ one-half unit to the left.

3. The graph of $y = a(x - h)^2$ where a and h are real numbers and $a \neq 0$ can be obtained by moving the graph of $y = ax^2$ $|h|$ units to the right if h is positive, and $|h|$ units to the left if h is negative.

Page 499.

$y = \frac{1}{2}(x - 3)^2 + 2$

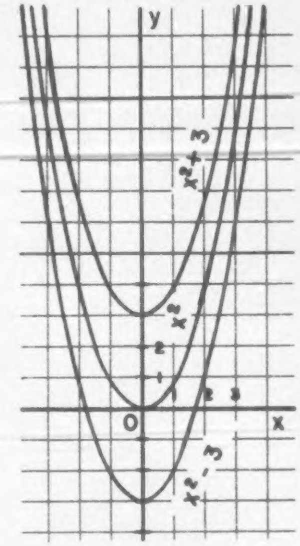
x	0	$\frac{1}{3}$	1	$\frac{3}{2}$	2	2.5	3	$\frac{13}{4}$	4	5
y	$\frac{13}{2}$	$\frac{50}{9}$	4	$\frac{25}{8}$	$\frac{5}{2}$	2.125	2	$\frac{65}{32}$	$\frac{5}{2}$	4

Page 500. The vertex of the parabola whose equation is $y = \frac{1}{2}(x - 3)^2 + 2$ is the point (3,2). Note that this is the point that the origin (0,0) goes to when all points of the plane are moved 3 units to the right and 2 units upward. The origin is the vertex of any parabola of the form ax^2.

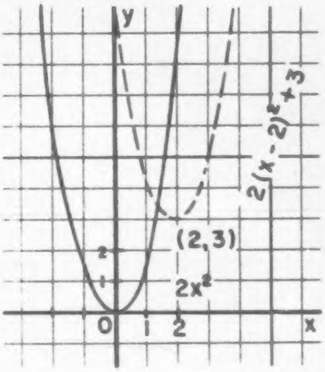
The axis of the parabola is the line whose equation is $x = 3$.

Answers to Problem Set 16-1d; page 501:

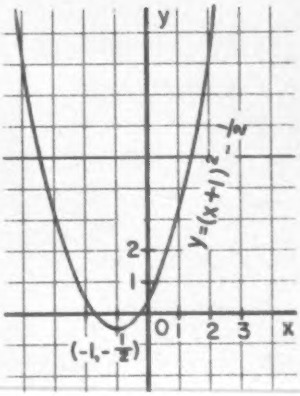
1. The graph of $x^2 - 3$ can be obtained by sliding the graph of x^2 down 3 units. The graph of $x^2 + 3$ can be obtained by sliding the graph of x^2 up 3 units.



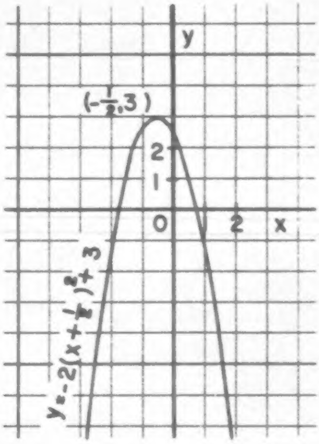
2. The graph of $2(x - 2)^2 + 3$ can be obtained by sliding the graph of $2x^2$ to the right two units and up three units.



3. The graph of $y = (x + 1)^2 - \frac{1}{2}$ can be obtained by sliding the graph of x^2 to the left one unit and down one-half unit. Its vertex is $(-1, -\frac{1}{2})$. The equation of its axis is $x = -1$.



4. The graph of $y = -2(x + \frac{1}{2})^2 + 3$ can be obtained by sliding the graph of $-2x^2$ to the left one-half unit and up three units.



5. (a) $y = (x + 5)^2 - 2$.
   (b) $y = -(x + 2)^2 + 3$.
   (c) $y = \frac{1}{3}(x - \frac{1}{2})^2 - 1$.
   (d) $y = \frac{1}{2}x^2$.

6. (a) The graph of $y = 3x^2$ is moved two units to the right and four units downward.
   (b) The graph of $y = -x^2$ is moved three units to the left and one unit upward.

[page 501]

(c) The graph of $y = \frac{1}{2}x^2$ is moved two units to the right and two units downward.

(d) The graph of $y = -2x^2$ is moved one unit to the left and two units upard.

16-2. Standard Forms

We discuss in detail and provide practice in changing quadratic polynomials to the standard form $a(x - h)^2 + k$ for two reasons. First, we have already shown how quickly the graph may be sketched from this form. Second, we shall show its application to the solving of quadratic equations.

Answers to Problem Set 16-2; pages 503-504:

1. (a) $x^2 - 2x = (x^2 - 2x + 1) - 1$
$= (x - 1)^2 - 1$

(b) $x^2 + x + 1 = (x^2 + x + \frac{1}{4}) + 1 - \frac{1}{4}$
$= (x + \frac{1}{2})^2 + \frac{3}{4}$

(c) $x^2 + 6x = (x^2 + 6x + 9) - 9$
$= (x + 3)^2 - 9$

(d) $x^2 - 3x - 2 = (x^2 - 3x + \frac{9}{4}) - 2 - \frac{9}{4}$
$= (x - \frac{3}{2})^2 - \frac{17}{4}$

(e) $x^2 - 3x + 2 = (x^2 - 3x + \frac{9}{4}) + 2 - \frac{9}{4}$
$= (x - \frac{3}{2})^2 - \frac{1}{4}$

(f) $5x^2 - 10x - 5 = 5(x^2 - 2x + 1) - 5 - 5(1)$
$= 5(x - 1)^2 - 10$

(g) $4x^2 + 4$ (or: $4(x - 0)^2 + 4$)

(h) $$x^2 + kx = \left(x^2 + kx + \frac{k^2}{4}\right) - \frac{k^2}{4}$$
$$= \left(x + \frac{k}{2}\right)^2 - \frac{k^2}{4}$$

(i) $$x^2 + \sqrt{2}\,x - 1 = \left(x^2 + \sqrt{2}\,x + \frac{2}{4}\right) - 1 - \frac{1}{2}$$
$$= \left(x + \frac{\sqrt{2}}{2}\right)^2 - \frac{3}{2}$$

(j) $$\frac{1}{2}x^2 - 3x + 2 = \frac{1}{2}(x^2 - 6x + 9) + 2 - \frac{9}{2}$$
$$= \frac{1}{2}(x - 3)^2 - \frac{5}{2}$$

2. (a) $$x^2 - x + 2 = \left(x^2 - x + \frac{1}{4}\right) + 2 - \frac{1}{4}$$
$$= \left(x - \frac{1}{2}\right)^2 + \frac{7}{4}$$

(b) $$x^2 + 3x + 1 = \left(x^2 + 3x + \frac{9}{4}\right) + 1 - \frac{9}{4}$$
$$= \left(x + \frac{3}{2}\right)^2 - \frac{5}{4}$$

(c) $$3x^2 - 2x = 3\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) - \frac{1}{3}$$
$$= 3\left(x - \frac{1}{3}\right)^2 - \frac{1}{3}$$

(d) $(x + 5)(x - 5) = x^2 - 25$ (or: $(x - 0)^2 - 25$)

(e) $$6x^2 - x - 15 = 6\left(x^2 - \frac{1}{6}x + \frac{1}{144}\right) - 15 - \frac{1}{24}$$
$$= 6\left(x - \frac{1}{12}\right)^2 - 15\frac{1}{24}$$

(f) $(x + 1 - \sqrt{2})(x + 1 + \sqrt{2}) = (x + 1)^2 - 2$

3. (a) The graph of $x^2 - x + 2$ is a parabola with vertex $(\frac{1}{2}, \frac{7}{4})$ and axis $x = \frac{1}{2}$. It is obtained by moving the graph of $y = x^2$ to the right $\frac{1}{2}$ unit and upward $1\frac{3}{4}$ units.

(b) The graph of $x^2 + 3x + 1$ is a parabola with vertex $(-\frac{3}{2}, -\frac{5}{4})$ and axis $x = -\frac{3}{2}$. It is obtained by moving the graph of $y = x^2$ to the left $1\frac{1}{2}$ units and downward $1\frac{1}{4}$ units.

(c) The graph of $3x^2 - 2x$ is a parabola with vertex $(\frac{1}{3}, -\frac{1}{3})$ and axis $x = \frac{1}{3}$. It is obtained by moving the graph of $y = 3x^2$ to the right $\frac{1}{3}$ unit and downward $\frac{1}{3}$ unit.

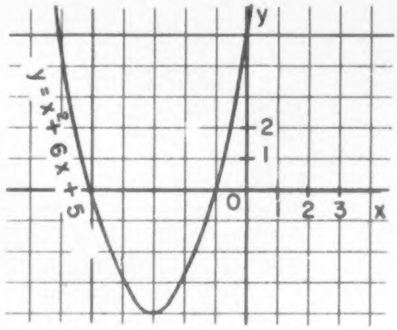
(d) The graph of $(x + 5)(x - 5)$ is a parabola with vertex $(0, -25)$ and axis $x = 0$. It is obtained by moving the graph of $y = x^2$ downward 25 units.

(e) The graph of $6x^2 - x - 15$ is a parabola with vertex $(\frac{1}{12}, -15\frac{1}{24})$ and axis $x = \frac{1}{12}$. It is obtained by moving the graph of $y = 6x^2$ to the right $\frac{1}{12}$ unit and downward $15\frac{1}{24}$ units.

(f) The graph of $(x + 1 - \sqrt{2})(x + 1 + \sqrt{2})$ is a parabola with vertex $(-1, -2)$ and axis $x = -1$. It is obtained by moving the graph of $y = x^2$ to the left one unit and downward two units.

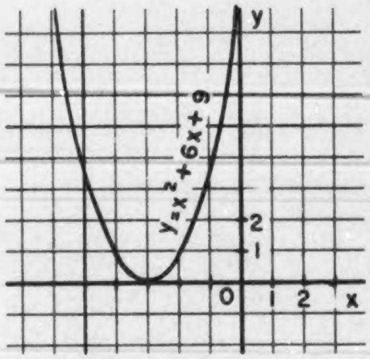
4. $$x^2 + 6x + 5 = (x^2 + 6x + 9) + 5 - 9$$
$$= (x + 3)^2 - 4$$

The graph crosses the x-axis in two points. These are $(-1,0)$ and $(-5, 0)$.



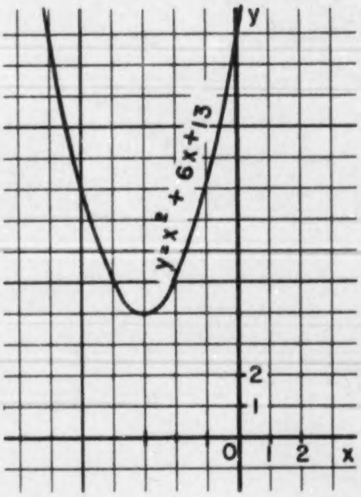
5. $x^2 + 6x + 9 = (x + 3)^2$

The graph does not cross the x-axis. It touches it at one point: (-3, 0).



6. $x^2 + 6x + 13 = (x^2 + 6x + 9) + 13 - 9$
$= (x + 3)^2 + 4$

The graph does not cross, nor touch, the x-axis.



7. If $y = 0$ in Problem 4, then

$$x^2 + 6x + 5 = 0$$

$$(x + 5)(x + 1) = 0$$

$$x + 5 = 0 \quad \text{or} \quad x + 1 = 0$$

$$x = -5, \quad \text{or} \quad x = -1$$

are all equivalent. Hence the truth set of the equation Is $\{-5, -1\}$.

If $y = 0$ in Problem 5, then

$$x^2 + 6x + 9 = 0$$

$$(x + 3)(x + 3) = 0$$

$$x + 3 = 0 \quad \text{or} \quad x + 3 = 0$$

$$x = -3 \quad \text{or} \quad x = -3$$

are all equivalent. Hence the truth set of the equation is $\{-3\}$.

In each case, the members of the truth set of the equation are the same as the abscissas of the points where the parabola crosses the x-axis.

The points in which a parabola crosses the x-axis will be those points whose abscissas are members of the truth set of the equation of the parabola, and whose ordinates are 0.

8.

	Polynomial	Standard form
Problem 4.	$x^2 + 6x + 5$	$(x + 3)^2 - 4$
Problem 5.	$x^2 + 6x + 9$	$(x + 3)^2 - 0$
Problem 6.	$x^2 + 6x + 13$	$(x - 3)^2 + 4$

The polynomial in Problem 4 can be factored as the difference of two squares, as follows:

$$\begin{aligned} x^2 + 6x + 5 &= (x + 3)^2 - 4 \\ &= (x + 3)^2 - (2)^2 \\ &= (x + 3 + 2)(x + 3 - 2) \\ &= (x + 5)(x + 1) \end{aligned}$$

16-3. Quadratic Equations

Page 505. $2x^2 - 3x + 1 = 0$

$$(2x - 1)(x - 1) = 0$$

$$2x - 1 = 0 \quad \text{or} \quad x - 1 = 0$$

$$x = \frac{1}{2} \quad \text{or} \quad x = 1$$

are all equivalent. Hence the truth set of the equation is $\{\frac{1}{2}, 1\}$.

Page 506. For every real number x, $x-1$ is also real. Hence $(x - 1)^2$ is greater than or equal to 0, since the square of a real number cannot be negative.

Answers to Problem Set 16-3; pages 507-509:

1. (a) $(3x - 5)(2x + 3)$

   (b) Cannot be factored over the real numbers.

   (c) $(x^2 + 8x + 16) - 16 + 3 = (x + 4)^2 - 13$

   $$= (x + 4 + \sqrt{13})(x + 4 - \sqrt{13})$$

(d) Cannot be factored over the real numbers.

(e) $(x + \sqrt{3})(x - \sqrt{3})$

(f) $(3x - 2)(3x - 2)$

(g) $(\sqrt{2}(x - 1) + \sqrt{5})(\sqrt{2}(x - 1) - \sqrt{5})$

$= (\sqrt{2}\,x - \sqrt{2} + \sqrt{5})(\sqrt{2}\,x - \sqrt{2} - \sqrt{5})$

(h) $x(3 - 2x)$

2. (a) $x^2 + 6x + 4 = 0$

$$(x^2 + 6x + 9) + 4 - 9 = 0$$

$$(x + 3)^2 - 5 = 0$$

$$(x + 3 + \sqrt{5})(x + 3 - \sqrt{5}) = 0$$

$$x + 3 + \sqrt{5} = 0 \quad \text{or} \quad x + 3 - \sqrt{5} = 0$$

$$x = -3 - \sqrt{5} \quad \text{or} \quad x = -3 + \sqrt{5}$$

are all equivalent. Hence the truth set of the equation is $\{-3 - \sqrt{5}, -3 + \sqrt{5}\}$.

(b)

$$2x^2 - 5x = 12$$

$$2x^2 - 5x - 12 = 0$$

$$2(x^2 - \frac{5}{2}x + \frac{25}{16}) - 12 - \frac{50}{16} = 0$$

$$2(x - \frac{5}{4})^2 - \frac{242}{16} = 0$$

$$(x - \frac{5}{4})^2 - \frac{121}{16} = 0$$

$$x - \frac{5}{4} + \frac{11}{4} = 0 \quad \text{or} \quad x - \frac{5}{4} - \frac{11}{4} = 0$$

$$x = -\frac{3}{2} \quad \text{or} \quad x = 4$$

are all equivalent. Hence the truth set of the equation is $\{-\frac{3}{2}, 4\}$. Note that in this problem we could write $2x^2 - 5x - 12 = (2x + 3)(x - 4)$ and obtain the truth set from this factored form.

[page 507]

(c)

$$x^2 + 4x + 6 = 0$$

$$(x^2 + 4x + 4) + 6 - 4 = 0$$

$$(x + 2)^2 + 2 = 0$$

are all equivalent. However $(x + 2)^2 + 2$ cannot be factored over the real numbers, and the truth set of the equation is $\emptyset$.

(d)

$$x^2 = 2x + 4$$

$$x^2 - 2x - 4 = 0$$

$$(x^2 - 2x + 1) - 4 - 1 = 0$$

$$(x - 1)^2 - 5 = 0$$

$$(x - 1 + \sqrt{5})(x - 1 - \sqrt{5}) = 0$$

$$x - 1 + \sqrt{5} = 0 \quad \text{or} \quad x - 1 - \sqrt{5} = 0$$

$$x = 1 - \sqrt{5} \quad \text{or} \quad x = 1 + \sqrt{5}$$

are all equivalent. Hence the truth set of the equation is $\{1 - \sqrt{5}, 1 + \sqrt{5}\}$.

(e)

$$2x^2 = 4x - 11$$

$$2x^2 - 4x + 11 = 0$$

$$2(x^2 - 2x + 1) + 11 - 2 = 0$$

$$2(x - 1)^2 + 9 = 0$$

are all equivalent. However, $2(x - 1)^2 + 9$ cannot be factored over the real numbers. Hence the truth set of the equation is $\emptyset$.

(f)

$$12x^2 - 8x = 15$$

$$12x^2 - 8x - 15 = 0$$

$$12(x^2 - \frac{2}{3}x + \frac{1}{9}) - 15 - \frac{12}{9} = 0$$

$$12(x - \frac{1}{3})^2 - \frac{147}{9} = 0$$

$$4(x - \frac{1}{3})^2 - \frac{49}{9} = 0$$

$$(2(x - \frac{1}{3}) + \frac{7}{3})(2(x - \frac{1}{3}) - \frac{7}{3}) = 0$$

$$2x + \frac{5}{3} = 0 \quad \text{or} \quad 2x - \frac{9}{3} = 0$$

$$x = -\frac{5}{6} \text{ or } x = \frac{3}{2}$$

are all equivalent. Hence the truth set of the equation is $\{-\frac{5}{6}, \frac{3}{2}\}$. (Note that $12x^2 - 8x - 15$ can be factored as a polynomial over the integers.)

3.

$$\begin{aligned} -3x^2 + 6x - 5 &= -3(x^2 - 2x) - 5 \\ &= -3(x^2 - 2x + 1) - 5 - (-3) \\ &= -3(x - 1)^2 - 2 \end{aligned}$$

From the equation in standard form we can see that the vertex of the parabola is the point $(1, -2)$.

Since the coefficient of $(x - 1)^2$ is negative, the parabola opens downward, and -2 is the largest value the polynomial $-3x^2 + 6x - 5$ can have.

4.

$$\begin{aligned} x^2 - 3x + 21 &= (x^2 - 8x) + 21 \\ &= (x^2 - 8x + 16) + 21 - 16 \\ &= (x - 4)^2 + 5 \end{aligned}$$

The vertex of the parabola which is the graph of the polynomial is the point $(4, 5)$. Since this parabola opens upward, the least value of the polynomial is 5.

It may have many values greater than 5.

Its values are integral for integral values of x, but not for all real values of x.

5.
$$2x^2 - 4x - 1 = 0$$
$$2(x - 1)^2 - 3 = 0$$
$$(\sqrt{2}(x - 1) + \sqrt{3})(\sqrt{2}(x - 1) - \sqrt{3}) = 0$$
$$\sqrt{2}\,x - \sqrt{2} + \sqrt{3} = 0 \quad \text{or} \quad \sqrt{2}\,x - \sqrt{2} - \sqrt{3} = 0$$
$$x = \frac{\sqrt{2} - \sqrt{3}}{\sqrt{2}} \quad \text{or} \quad x = \frac{\sqrt{2} + \sqrt{3}}{\sqrt{2}}$$

are all equivalent. Hence the truth set of the equation is $\{\frac{\sqrt{2} - \sqrt{3}}{\sqrt{2}}, \frac{\sqrt{2} + \sqrt{3}}{\sqrt{2}}\}$.

In Chapter 11 we learned to rationalize the denominator; hence we see that the truth set can also be written $\{1 - \frac{\sqrt{6}}{2}, 1 + \frac{\sqrt{6}}{2}\}$.

6. If the number of feet in the width is x and the number of feet in the perimeter is 94 then the number of feet in the length is 47 - x. The number of square feet in the area is x(47 - x).

$$x(47 - x) = 496$$
$$x^2 - 47x + 496 = 0$$
$$(x^2 - 47x + \frac{2209}{4}) + 496 - \frac{2209}{4} = 0$$
$$(x - \frac{47}{2})^2 - \frac{225}{4} = 0$$
$$(x - \frac{47}{2} + \frac{15}{2})(x - \frac{47}{2} - \frac{15}{2}) = 0$$
$$(x - 16)(x - 31) = 0$$
$$x - 16 = 0 \quad \text{or} \quad x - 31 = 0$$
$$x = 16 \quad \text{or} \quad x = 31$$

are all equivalent. Hence the truth set of the equation is is $\{16, 31\}$. When $x = 16$, $47 - x = 31$. The width is 16 ft., and the length is 31 ft.

As you see $x^2 - 47x + 496$ is factorable over the integers so it would not have been necessary to use the method of completing the square in this case.

The student might try to do this problem using two variables:

If the rectangle is x feet wide and y feet long, then

$$\begin{cases} 2x + 2y = 94 \\ xy = 496 \end{cases}$$

$$\begin{cases} x + y = 47 \\ xy = 496 \end{cases}$$

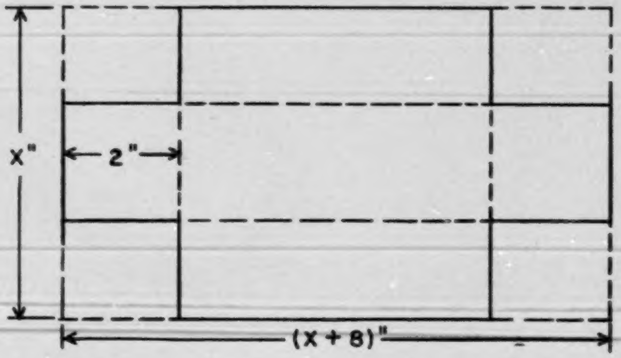
$$\begin{cases} y = 47 - x \\ xy = 496 \end{cases}$$

$$\begin{cases} y = 47 - x \\ x(47 - x) = 496 \end{cases}$$

Although this involves a quadratic equation in two variables, the student may see that the "substitution" method reduces it to an equation in one variable.

7. If the sheet of metal is x inches wide, it is $x + 8$ inches long.

The box is $x - 4$ inches wide, $x + 8 - 4$ inches long, and 2 inches deep.



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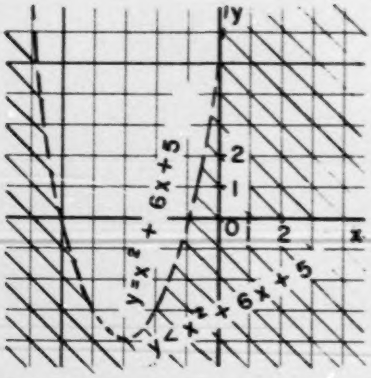
$$2(x - 4)(x + 4) = 256 \quad \text{and} \quad x > 0$$
$$2(x^2 - 16) = 256 \quad \text{and} \quad x > 0$$
$$x^2 - 16 = 128 \quad \text{and} \quad x > 0$$
$$x^2 - 144 = 0 \quad \text{and} \quad x > 0$$
$$(x + 12)(x - 12) = 0 \quad \text{and} \quad x > 0$$
$$x + 12 = 0 \quad \text{or} \quad x - 12 = 0, \quad \text{and} \quad x > 0$$
$$x = -12 \quad \text{or} \quad x = 12, \text{ and} \quad x > 0$$

are all equivalent. Hence the truth set of the sentence is {12}.

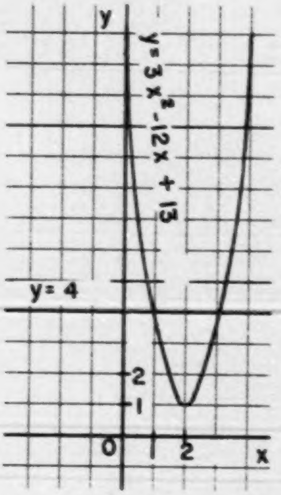
The sheet of metal is 12 inches wide and 20 inches long.

8.

(a)

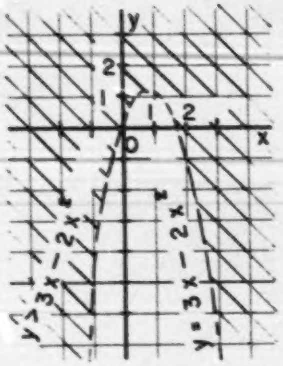


(b)

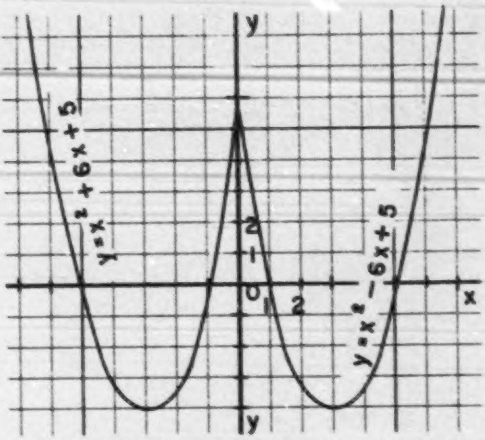


The graph is the two points (1,4) and (3,4).

(c)



(d)



(d) We recall here that $y = x^2 - 6|x| + 5$ implies:

$$y = \begin{cases} x^2 - 6x + 5, & x \geq 0 \\ x^2 + 6x + 5, & x < 0. \end{cases}$$

9. If one leg is y feet long, the other leg is $y - 1$ feet long, and the hypotenuse is $y + 8$ feet long.

$$y^2 + (y - 1)^2 = (y + 8)^2 \text{ and } y > 0$$

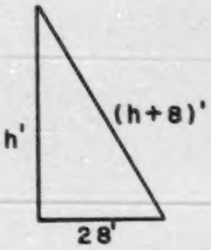
Truth set: $\{21\}$.
The lengths of the sides of the triangle are 20 feet, 21 feet, and 28 feet.

10. If the window is h feet above the ground, the rope is $h + 8$ feet long.

$h^2 + 28^2 = (h + 8)^2$ and $h > 0$.

Truth set: $\{45\}$.

The window is 45 feet above the ground.



11. If a leg of the triangle is x feet long.

$$x^2 + x^2 = 3^2 \text{ and } x > 0$$

Truth set: $\{\frac{3\sqrt{2}}{2}\}$

Each leg is $\frac{3\sqrt{2}}{2}$ units long.

12. If the diagonal is d inches long, a side is $d - 2$ inches long.

$$(d - 2)^2 + (d - 2)^2 = d^2 \text{ and } d > 0$$

Truth set: $\{4 + 2\sqrt{2}\}$.

The diagonal is $4 + 2\sqrt{2}$ inches long.

13. If the sheet is t feet long, the width is $t-3$ feet.

$$t(t - 3) = 46\frac{3}{4} \text{ and } t > 0$$

Truth set: $\{\frac{17}{2}\}$

The sheet is $8\frac{1}{2}$ feet long.

14. If n is one of the numbers, $9 - n$ is the other number.

$$n^2 - (9 - n)^2 = 25$$

Truth set: $\{\frac{53}{9}\}$

The numbers are $\frac{53}{9}$ and $\frac{28}{9}$.

15. If n is the number

$$14n + n^2 = 11$$

Truth set: $\{-7 + 2\sqrt{15}, -7 - 2\sqrt{15}\}$

The number is $(-7 + 2\sqrt{15})$ or $(-7 - 2\sqrt{15})$.

16. If his average speed going was g miles per hour, then his average speed returning was $g - 6$ miles per hour.

$$\frac{336}{g-6} = \frac{336}{g} + 1 \qquad g > 0$$

Truth set: $\{48\}$

The average speed was 48 miles per hour going and 42 miles per hour returning.

17. If x is the number,

$$x + \frac{1}{x} = 4 \quad \text{and} \quad x \neq 0.$$

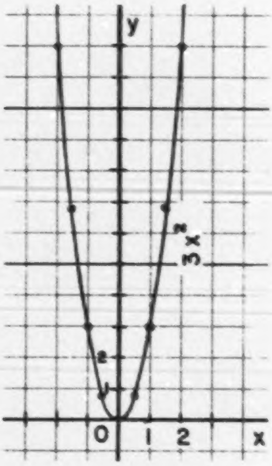
Truth set: $\{2 + \sqrt{3},\ 2 - \sqrt{3}\}$

The number is either $2 + \sqrt{3}$ or $2 - \sqrt{3}$

Answers to Review Problems; pages 509-510:

1. (a)

x	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
y	12	$\frac{27}{4}$	3	$\frac{3}{4}$	0	$\frac{3}{4}$	3	$\frac{27}{4}$	12



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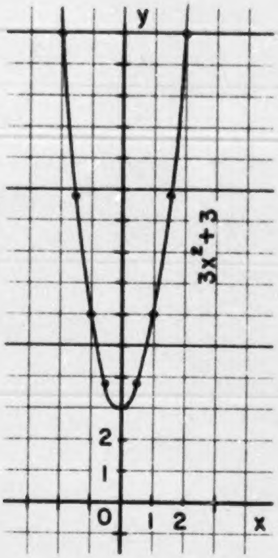
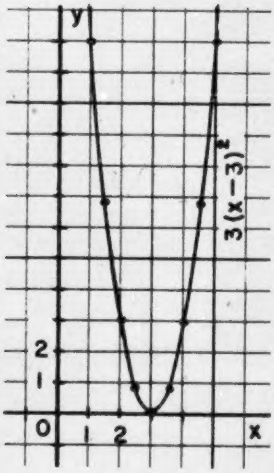
(b)



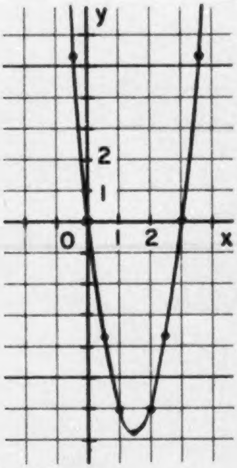
(c)



(d) $3x(x - 3) = 3x^2 - 9x$

$$= 3(x^2 - 3x + \frac{9}{4}) - \frac{27}{4}$$

$$= 3(x - \frac{3}{2})^2 - \frac{27}{4}$$



(e) The graph of (d) can be obtained by moving the graph of (a) $\frac{3}{2}$ units to the right and $\frac{27}{4}$ units down.

2. (a) $y = -x^2$

(b) $y = x^2 + 3$

(c) $y = (x + 2)^2$

(d) $y = (x - 1)^2 - 2$

3. (a) Line
   (b) Parabola
   (c) Pair of lines: $y = x$ or $y = -x$
   (d) Line
   (e) Line
   (f) Parabola
   (g) Parabola
   (h) Line
   (i) Point

4. (a)

(i) The graph crosses the y-axis at a point where $x = 0$. When $x = 0$, $x^2 + 2x - 8 = -8$. The graph crosses the y-axis at -8.

(ii) The graph crosses the x-axis at points where

$$x^2 + 2x - 8 = 0$$

$$(x - 2)(x + 4) = 0$$

$$x - 2 = 0 \quad \text{or} \quad x + 4 = 0$$

$$x = 2 \quad \text{or} \quad x = -4$$

The graph crosses the x-axis at 2 and at -4.

(iii) The largest or smallest value of the polynomial is the the ordinate of the vertex of the parabola.

$$x^2 + 2x - 8$$

$$(x^2 + 2x + 1) - 1 - 8$$

$$(x - 1)^2 - 9$$

The vertex is at $(1, -9)$ and the parabola opens upward.

The smallest value is -9.

(b) (i) 3; (ii) Does not cross the x-axis; (iii) Smallest value at 2.

(c) (i) 4; (ii) 2 and -2; (iii) Largest value at 4.

(d) (i) 4; (ii) 4; No largest or smallest value. The graph is a line.

(e) (i) 12; (ii) 4 and -3; (iii) Largest value at $\frac{49}{4}$.

(f) (i) 0; (ii) 0; (iii) Smallest value at 0.

5. (a) $\{7, 3\}$

(b) $\{1 + \sqrt{2}, 1 - \sqrt{2}\}$

(c) $\{\frac{1}{3}, -\frac{1}{2}\}$

(d) $\{\frac{3}{5}, \frac{6}{7}\}$

(e) $\{2 + \sqrt{3}, 2 - \sqrt{3}\}$

(f) The set of all real numbers.

6. If one of the numbers is n,
the other is $9 - n$, and
their product is $n(9 - n)$.

$$\begin{aligned} n(9 - n) &= 9n - n^2 \\ &= -(n^2 - 9n + \frac{81}{4}) + \frac{81}{4} \\ &= -(n - \frac{9}{2})^2 + \frac{81}{4} \end{aligned}$$

The vertex of the parabola is $(\frac{9}{2}, \frac{81}{4})$ and it opens downward. Hence, the value of n which makes $n(9 - n)$ largest is $\frac{9}{2}$.

The numbers are $\frac{9}{2}$ and $\frac{9}{2}$.

7. $$\begin{aligned} n^2 - 10n + 175 &= (n^2 - 10n + 25) + 150 \\ &= (n - 5)^2 + 150 \end{aligned}$$

The vertex of the parabola is at $(5, 150)$.
He should manufacture 5 boats a day; this will result in a minimum cost of $150 per boat.

Chapter 16

Suggested Test Items

1. Draw a graph of $y = x^2$.

   (a) Explain how the graph of $y = -x^2$ can be obtained from the graph of $y = x^2$.

   (b) Explain how the graph of $y = x^2 + 6$ can be obtained from the graph of $y = x^2$.

2. Put each of the following in standard form and draw its graph.

   (a) $x^2 - 2x - 3$

   (b) $x(x + 6)$

   (c) $6 - 6x - x^2$

   (d) $2x^2 + 4x + 8$

   (e) $(2x + 1)(2x - 5)$

   (f) $(x - 3)(x + 3)$

3. Solve.

   (a) $x^2 + 5x + 4 = 0$

   (b) $4x^2 + 3 = 8x$

   (c) $2x^2 + 1 = 4x$

   (d) $x^2 - 2\sqrt{3}x + 1 = 0$

   (e) $x^2 - 2x + 2 = 0$

   (f) $x(x + 1) - (x + 1) = x^2 + 13$

4. The polynomial "$x^2 - 6x + 16$" may never have a value less than what positive integer?

5. Two successive prime numbers differ by 2 and their product is 899. Find the numbers.

6. Each dimension of an 8' by 12' room is increased the same amount. The floor space is increased 224 square feet. Find the length and width of the new room.

Chapter 17

FUNCTIONS

17-1. The Function Concept.

This chapter treats one of the most important and most basic ideas in mathematics - the idea of a function. It is included here for those selected classes of better students, as well as exceptional individuals, who are able to move ahead fast enough to complete the previous material in less than the expected number of lessons. The teacher can find additional discussion of this concept in *Studies in Mathematics*, Volume III, pages 6.17-6.25.

In the past it has been customary to postpone a careful study of functions to a much more advanced mathematical setting. Because of this, the subject is surrounded by an aurora of difficulty which is completely undeserved. The idea is simple and, as will become evident as we proceed, is involved implicitly in our most elementary considerations. In this respect the function concept is in the same category as the set concept. Incidentally, the set concept is another good example of an idea which was involved implicitly in many mathematical situations long before it was finally separated out and studied carefully in its own right. As yet another illustration of this phenomenon, we mention the general ACD properties of addition and multiplication which are implicit in all of arithmetic but are not made explicit until the study of algebra.

Page 512. The completed postage table is

ounces	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$2\frac{1}{4}$	$2\frac{1}{2}$	$2\frac{3}{4}$	3	$3\frac{1}{4}$	...
cents	4	4	4	4	8	8	8	8	12	12	12	12	16	...

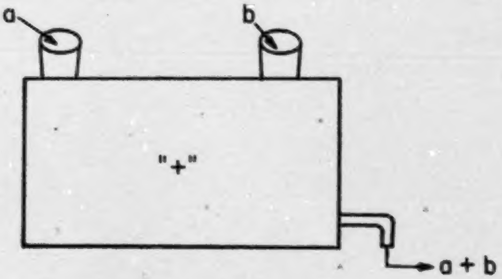
Answers to Problem Set 17 - 1a; pages 513-515:

1. (a) The association is *from* the set of all positive integers *to* all odd positive integers. Since we obviously cannot list all odd integers, the table can only *suggest* the full association. The odd integers associated with 13 and 1000 are 25 and 1999 respectively.

A rule for the association might be stated as "with each positive integer n associate the nth odd positive integer." Another way of giving this association is "with each positive integer n associate the integer 2n-1."

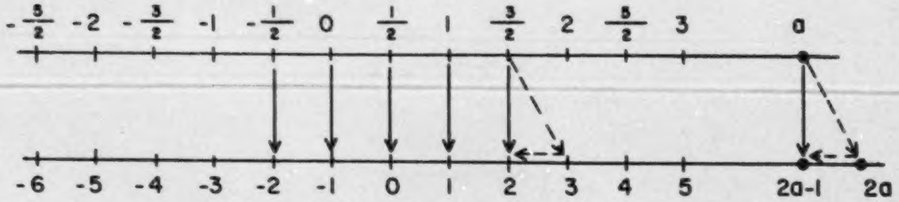
(b) The machine picture can only suggest the full association described here. The association is from the set of all positive real numbers to the set of all real numbers greater than -1. The machine will give the number 33 when fed the number 17. It will reject the numbers 0 and -1 since it is "constructed" to accept only positive real numbers.

Some teachers like to emphasize the machine idea much more than we have done here. If you are one of these, go ahead! The machine can also be used to visualize all of the algebraic operations. For example, an addition machine might be pictured as follows:



Notice that this suggests that the operation of addition can be regarded as a function which associates with each ordered pair of real numbers a real number.

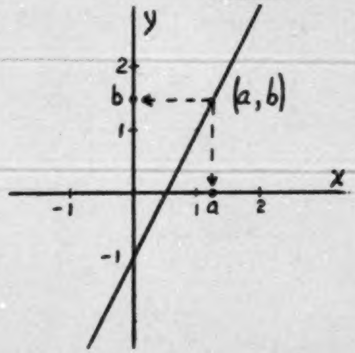
(c) The association here is from the set of all real numbers to the set of all real numbers. The association can be represented as follows:



In order to determine what number is associated with any real number a on the upper line, we observe first that, before the lower line is moved to the right (i.e. if 0 is directly under 0), the number on the lower line directly below a is 2a. This follows immediately from the assumption that the unit on the upper line is twice the unit on the lower line. Moving the lower number line one unit to the right from its original position has the effect of subtracting 1 from the original coordinate of each point on the lower line. Therefore the number associated with a is 2a-1. The number associated with -13 is -27, with 13 is 25 and with 1000 is 1999.

Note: It may not be worth the time that would be required to get across to all students the above general argument that 2a-1 is the number associated with a. The main purpose here is not to study this special way of setting up an association but rather to emphasize the fact that an association can arise in a variety of very different ways.

(d) This exercise is very important and should be mastered completely by everyone. The association is from the set of all real numbers to the set of all real numbers. Notice that to each number a there is exactly one number b such that (a,b) is a point on the line. This is what makes it possible to set up the association in this case. The number associated with -1 is -3, with $-\frac{1}{2}$ is -2 and with 13 is 25. Since the y-form of the equation of the line is $y = 2x - 1$, we conclude that the number associated with the number a is $2a-1$.



(e) This association is from the set of all real numbers t such that $-1 < t < 1$ to the set of all real numbers y such that $-3 < y < 1$. Associated with $-\frac{2}{3}$ is the number $-\frac{7}{3}$. Since $|2| > 1$, no number is associated with 2.

(f) The association is from the set of all negative real numbers to the set of all real numbers y such that $y < -1$. Again the number associated with the negative number a is $2a-1$. The number associated with -13 is -27. No number is associated with 0.

2. (a) The association is from the set of all real numbers x such that $x < 1$ to the set of all real numbers y such that $y < -3$. The rule is "To each real number less than 1 assign the number obtained by multiplying the given number by 2 and then subtracting 5." The rule assigns exactly one number to each number in the first set.

(b) The association is from the set of all non-negative real numbers to the set of all real numbers. The rule is "To each non-negative real number assign numbers whose absolute values are equal to the given number." The rule assigns 0 to 0 and, to each positive number, the given number and its opposite.

(c) The association is from the set of all real numbers to the set of all real numbers. The rule is "To each real number x assign the number obtained by multiplying the given number by 3 and then adding 7." The rule assigns exactly one number to each real number.

(d) The association is from the set of all integers to the set of all real numbers. The rule is "To each integer assign those real numbers less than the given integer." The rule assigns an infinite set of real numbers to each integer.

(e) The association is from the set of all non-negative rational numbers to the set of all real numbers whose squares are rational. The rule is "To each non-negative rational number assign those real numbers whose squares are equal to the given rational number." The rule assigns 0 to 0 and, to each positive rational, its two square roots.

3. To each real number between 0 and 320, associate 4 times the smallest integer which is greater than or equal to the given number.

4. The starting point here might be a machine such as the scales which print your weight on a card (along with your fortune). A student who has an idea of how such scales might be constructed should be able to suggest the modifications and

additions necessary for the purpose suggested here. The idea, of course, is to obtain another machine "picture" of an interesting association of numbers.

The definition of function which we have given is, strictly speaking, a definition of real function since we have restricted the domain and range to real numbers. In later courses, the student will meet more general types of functions in which the domain and range can be sets other than sets of real numbers. Such a function might, for example, have sets of points in the plane as its domain of definition. As an illustration, associate with each point (x,y) of the plane the abscissa x of the point. In this case the domain is the set of all points of the plane and the range is the set of all real numbers. If we associate with each point (x,y) of the plane the point $(-x,y)$ the result is a function with both domain and range equal to the set of all points in the plane.

In the discussion about functions, it is important to emphasize at every opportunity the following points:

(1) To each number in the domain of definition, the function assigns one and only one number from the range. In other words we do not have "multiple-valued" functions. However, the same number can be assigned to many different elements of the domain.

(2) The essential idea of function is found in the actual association from numbers in the domain to numbers in the range and not in the particular way in which the association happens to be described.

(3) Always speak of the association as being from the domain to the range. This helps fix the correct idea that we are dealing with an ordered pairing of numbers in which the number from the domain is mentioned first and the assigned number from the range is mentioned second.

(4) Not all functions can be represented by algebraic expressions.

Although the above points are not absolutely vital as far as elementary work with functions is concerned, they become of central importance later. Also, many of the difficulties which students have with the idea of function can be traced to confusion on these matters. Therefore it becomes important to make certain that the student understands these points from his very first contact with the function concept.

Answers to Problem Set 17-1b; pages 517-519:

1. All statements in Problem 1, and all in Problem 2 except (b), (d), (e) define functions. The statements in 2(b), (d), (e) fail to define functions since they associate more than one number with some of the numbers in the given set of real numbers.

2. (a) $2x - 5, x < 1$
   (b) No single expression
   (c) $3x + 7$, x any real number
   (d) No single expression
   (e) No single expression

3. (a) Domain of definition is the set $\{1,2,3,4,5,6,7,8,9,10\}$.

   (i)

Number of the day	1	2	3	4	5	6	7	8	9	10
Income for the day (dollars)	7	5	6	-4	-7	4	0	-3	-4	-6

   (ii) The function cannot be described by a simple algebraic expression in x. However it can be described by a polynomial of sufficiently high degree.

(b) The domain of definition is the set of all positive integers and the range is the set $\{0,1,2,3,4\}$.

(i)

Positive integer	1	2	3	4	5	6	7	8	9	10	11	...
Remainder after division by 5	1	2	3	4	0	1	2	3	4	0	1	...

(ii) The function cannot be represented by a simple algebraic expression, but we can do the following: With each positive integer n, associate

0, if $n = 5k$, k any positive integer,

1, if $n = 5k + 1$, k any positive integer.

2, if $n = 5k + 2$, k any positive integer.

3, if $n = 5k + 3$, k any positive integer.

4, if $n = 5k + 4$, k any positive integer.

(c) The domain of definition is the set of all positive real numbers.

(i)

positive real number a	$\frac{1}{2}$	1	$\sqrt{2}$	$\frac{2}{3}$	2	3	4	...
$\frac{1}{3}(a + 2)$	$\frac{5}{6}$	1	$\frac{1}{3}(\sqrt{2}+2)$	$\frac{8}{9}$	$\frac{4}{3}$	$\frac{5}{3}$	2	...

(ii) To each positive real number a associate the number $\frac{1}{3}(a + 2)$.

(d) The domain of definition is the set of all positive integers.

(i)

positive integer n	1	2	3	4	5	6	7	8	9	10	...
n^{th} prime	2	3	5	7	11	13	17	19	23	29	...

(ii) No function is known whose domain of definition is the positive integers, whose range is a set of primes, and whose rule is an algebraic expression.

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(e) Domain of definition is the set of all positive integers from 1 to 365 inclusive. The range is all non-negative integers from 0 to 364.

(i)

number of day	1	2	3	4	...	360	361	362	363	364	365
days remaining in year	364	363	362	361	...	5	4	3	2	1	0

(ii) To the n^{th} day associate $365-n$.

(f) Domain is the set of all positive integers less than or equal to the number of dollars which you possess.

(i)

number of dollars invested	1	2	3	...	20	...	100	...
interest at 6 %	0.06	0.12	0.18	...	1.20	...	6.00	...

(ii) If P is the number of dollars invested, associate $0.06P$.

(g) Domain is the set of all positive real numbers.

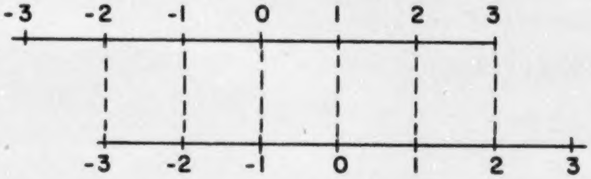
(i)

diameter in inches	$\frac{1}{4}$	$\frac{1}{\pi}$	$\frac{1}{2}$	$\frac{2}{\pi}$	$\sqrt{2}$	2
circumference in inches	$\frac{\pi}{4}$	1	$\frac{\pi}{2}$	2	$\sqrt{2}\pi$	2π

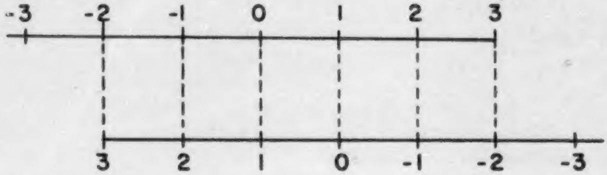
(ii) If d is the diameter, the circumference is πd.

(h) Domain is the set of all real numbers.

First position:



Second position:



(i)

real number a	-2	-1	0	$\frac{1}{2}$	1	2	10	...
number associated with a	3	2	1	$\frac{1}{2}$	0	-1	-9	...

(ii) Let a be any real number on the upper line. After the lower line is moved so that its point 0 is directly under the point 1 of the upper line, the point on the lower line directly under a is a - 1. When the lower line is rotated about 0, the number a - 1 is replaced by its opposite, -(a - 1) or 1 - a. Therefore the resulting association can be stated as follows:

To each real number a, associate the number 1 - a.

4. The domain is the set of all positive integers greater than 1.

positive integer n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
smallest factor in n greater than 1	2	3	2	5	2	7	2	3	2	11	2	13	2	3	...

The prime numbers are associated with themselves. The range is the set of all primes.

5. This function is similar to the example discussed at the beginning of the chapter. The domain is all real numbers between 0 and 32 and the range is all integers from 1 to 32. The function cannot be represented by an algebraic expression in one variable. The numbers assigned to 3.7 and 5 are 4 and 5, respectively.

6. The verbal description is the only method we can give now for representing this function. The numbers assigned to $-\pi$, $-\frac{3}{2}$, $\frac{\sqrt{2}}{2}$, 0, $\frac{1}{2}$, $\sqrt{2}$, $\frac{\pi}{2}$ and 10^6 are $1, -1, 1, -1, -1, 1, 1$ and -1, respectively.

7. Notice that $\frac{1}{x + 2}$ gives a real number for every value of x different from -2. Since we only have square roots of non-negative real numbers, $\sqrt{x + 2}$ is meaningful only for $x + 2 \geq 0$ or $x \geq -2$.

   (a) All real numbers except 3.

   (b) All real numbers greater than or equal to 1.

   (c) All real numbers except 0.

   (d) All real numbers.

   (e) All real numbers x such that $x^2 \geq 1$. This is the same as the set of all real numbers x such that either $x \geq 1$ or $x \leq -1$.

   (f) All real numbers except 2 and -2.

8. Notice that if the perimeter of a rectangle is equal to 10 ft. then the length of a side must be less than 5 ft.

   (a) The domain of definition here is the set of all positive integers between 0 and the number of dollars which you possess. (Some might want to think of borrowing as negative investment, in which case the domain would include negative integers depending on your credit.)

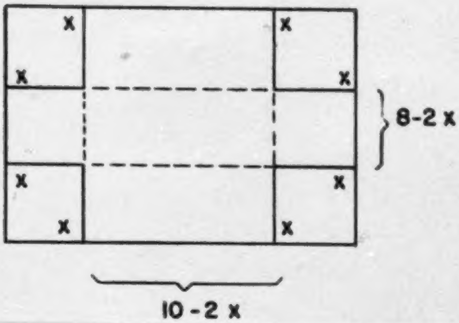
(b) Since the area of a triangle is equal to one-half the base x times the altitude a, we must have $\frac{1}{2}xa = 12$ or $a = \frac{24}{x}$. In this case the base can have any length whatsoever except 0. Therefore the domain is the set of all positive real numbers.

(c) The bottom of the box has dimensions $8-2x$ by $10-2x$. Hence the volume is given by

$$(8-2x)(10-2x)x$$

or

$$4x(4-x)(5-x).$$



Obviously the square which is removed must have its side less than 4, so that $0 < x < 4$. In other words the domain of definition is the set of all real numbers between 0 and 4.

17-2. The Function Notation

The function notation must be handled with great care. In the beginning one can not do too many examples and exercises of the type, "What is the value of f at 2?" or "What number is represented by $f(2)$?" Check the students on this at every opportunity. It is essential for everyone to understand that the symbol "f", when used for a function, stands for the complete function and not just for the rule or some special way of representing the function.

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Notice that we have avoided using the misleading expression "the function $f(x)$" as a substitute for the correct expression "the function f". In other words, $f(x)$ is a number which is the value of f at x, not the function itself.

If two variables x and y are related by the sentence $y = f(x)$, where f is a given function, then x is sometimes called the independent variable and y the dependent variable in the relation. This terminology, which is used by many, has been avoided here since it is so easily abused. It leads to expressions such as "y is a function of x" which tend to obscure the function concept.

For the function f defined by the rule,

$$f(x) = 2x - 1, \text{ for each real number } x,$$

$f(-\frac{4}{3}) = -\frac{11}{3}$, $f(-\frac{1}{2}) = -2$, $f(\frac{4}{3}) = \frac{5}{3}$, $f(s) = 2s-1$, $f(-t) = -2t - 1$, $-f(t) = -2t + 1$, $f(t - 1) = 2(t - 1) - 1 = 2t - 3$, $f(t) - 1 = (2t - 1) - 1 = 2t - 2$.

For the function g defined on page 521, the range is the set $\{-1,0,1\}$. Also, $g(-3.2) = -1$, $g(0) = 0$, $g(-\frac{1}{2}) = -1$, $g(\sqrt{2}) = 1$. If $a > 0$, $g(a) = 1$ and $g(-a) = -1$. If $a \neq 0$, $g(|a|) = 1$. The rule for g cannot be given by a simple algebraic expression in one variable.

Answers to Problem Set 17-2; pages 521-524:

1. (a) $F(-2) = 3$
   (b) $-F(2) = -1$
   (c) $F(-\frac{1}{2}) = \frac{9}{4}$
   (d) $F(1)-1 = \frac{1}{2}$
   (e) $F(0) = 2$
   (f) $|F(-6)| = 5$
   (g) $F(|-6|) = -1$
   (h) $F(t) = 2 - \frac{t}{2}$
   (i) $F(\frac{t}{2}) = 2 - \frac{t}{4}$
   (j) $F(2t) = 2-t$
   (k) $F(\frac{1}{t}) = 2 - \frac{1}{2t}$

2. The domain of definition of G is all real numbers and the range is all non-negative real numbers.

   (a) $G(0) = 0$ (b) $G(a) - G(-a) = 0$ (c) $\frac{G(-3)}{3} = 1$

3. The function h is identical with the function g defined in Section 17-2. Be sure that the students understand why $h = g$. Emphasize again that the function is independent of the particular method of describing it or the symbols used to represent it.

4. If $x < 0$, then $|x| = -x$, so that $\frac{x}{|x|} = -1$.

   If $x > 0$, then $|x| = x$, so that $\frac{x}{|x|} = 1$.

   Therefore

$$k(x) = \begin{cases} -1 , & x < 0 \\ 0 , & x = 0 \\ 1 , & x < 0 , \end{cases}$$

   and hence $k = g$.

5. (a) $H(2) = 3$ (b) $H(\frac{1}{3}) = -\frac{8}{9}$ (c) $H(-\frac{1}{3}) = -\frac{8}{9}$

   (d) $-H(-2) = -3$ (e) $H(-1) + 1 = 1$ (f) $H(3)$ is not defined.

   (g) $H(a) = a^2 - 1$, for $-3 < a < 3$.

   (h) $H(t-1) = t^2 - 2t$, for $-2 < t < 4$. Notice that,

   if $-2 < t < 4$, then $-3 < t-1 < 3$,

   so that $H(t-1)$ is defined (i.e. $t-1$ is in the domain of definition of H.)

   (i) $H(t) - 1 = t^2 - 2$, for $-3 < t < 3$.

6. (a) The domain of definition of Q is the set of all real numbers x such that $-1 \leq x < 0$ or $0 < x \leq 2$, i.e. all numbers between -1 and 2, except 0 and including -1 and 2.

(b) The range of Q consists of the number -1 along with all x such that $0 < x \leq 2$.

(c) $Q(-1) = -1$, $Q(-\frac{1}{2}) = -1$, $Q(0)$ is not defined, $Q(\frac{1}{2}) = \frac{1}{2}$, $Q(\frac{3}{2}) = \frac{3}{2}$, $Q(\pi)$ is not defined.

(d) R is the same function as Q. (See Problem 3 above.)

7. (a) $\{6\}$

(b) All x such that $4 < x$.

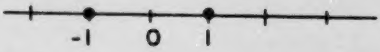
(c) $\{5\}$

(d) $\{\frac{4}{3}\}$

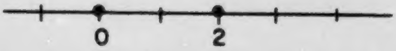
(e) All x such that $x < 0$.

(f) All x such that $2 \leq x$.

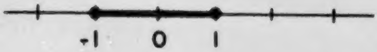
8. (a) $\{-1, 1\}$.



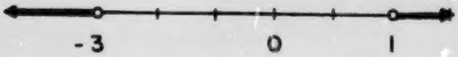
(b) $\{0, 2\}$.



(c) All x such that $-1 \leq x \leq 1$.



(d) All x such that either $x < -3$ or $x > 1$.



9. (a) The domain of definition of f is the set of all real numbers, and the domain of definition of F is the set of all real numbers different from -2. Therefore $f \neq F$. However, since

$$\frac{x^2 - 4}{x + 2} = x - 2, \text{ if } x \neq -2,$$

we have

$$f(x) = F(x), \text{ for all } x \neq -2.$$

(b) In both cases the domain of definition is the set of all real numbers. Since

$$\frac{t^4 - 1}{t^2 + 1} = t^2 - 1 \text{ for all real}$$

numbers t, it follows that $g = G$.

17-3. Graphs of Functions

The graph of a function gives us a quick way to picture certain properties of the function. In most cases we are primarily interested in the "shape" of the graph rather than in the precise location of individual points, although there may be certain key points which need to be located carefully and doubt about the shape of a portion of the graph can frequently be resolved by locating a few judiciously chosen points. Other kinds of information can also be helpful in determining the general shape of a graph. For example, without locating any points whatsoever, we know that the graph of $y = 3x^2 + 1$ must be above the line $y = 1$ since $3x^2 \geq 0$ for all x. Also, since $0 < a < b$ implies $0 < a^2 < b^2$, we know that the graph of $y = 3x^2 + 1$ rises to the right. It must be impressed on the student that the objective in drawing a graph is not simply to locate a lot of points but rather to discover the shape of the graph by any methods which can be applied.

The location of certain carefully chosen points of the graph is one of the methods.

Example 1. The graphs of the function f defined by:

$$f(x) = 2x - 1, \quad 0 \leq x < 2$$

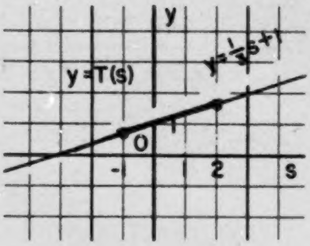
and the function F defined by:

$$F(x) = 2x - 1, \quad -2 < x < 2$$

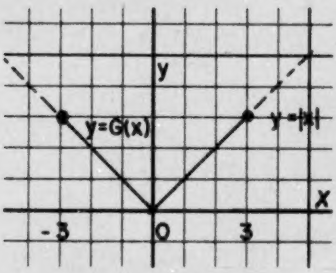
are different since the first graph is the line segment joining the points (0, -1) and (2, 3) with the first point included and the second excluded and the second graph is the line-segment joining the points (-2, -5) and (2, 3) with both end-points excluded.

Answers to Problem Set 17-3a; pages 525-526:

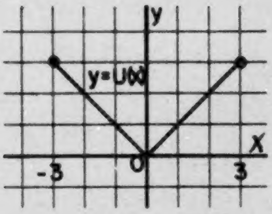
1. (a)



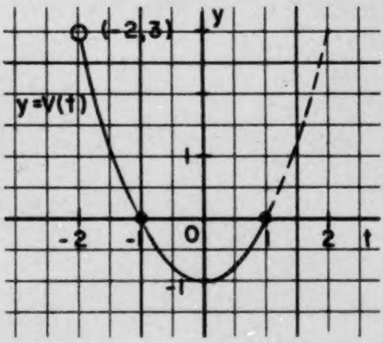
(b)



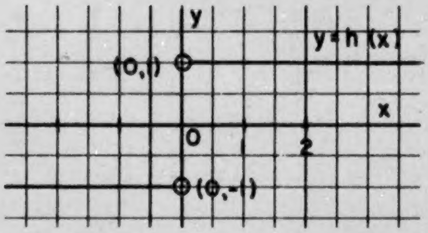
(c)



(d)



(e)



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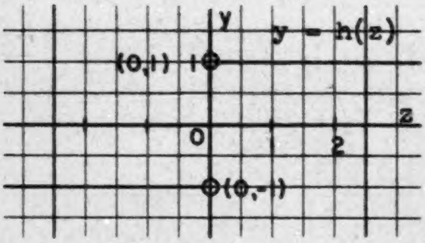
(f)



2. (a) Domain: all s such that $-1 \leq s \leq 2$. Range: all y such that $\frac{2}{3} \leq y \leq \frac{5}{3}$.

(b) Domain: all x such that $-3 \leq x \leq 3$. Range: all y such that $0 \leq y \leq 3$.

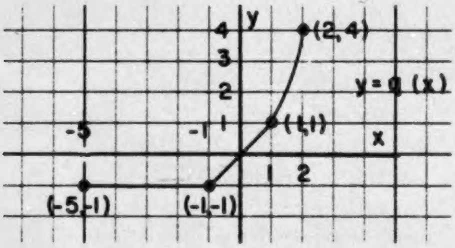
(c) Domain: all x such that $-3 \leq x < 3$. Range: all y such that $0 \leq y \leq 3$.

(d) Domain: all t such that $-2 < t \leq 1$. Range: all y such that $-1 \leq y < 3$.

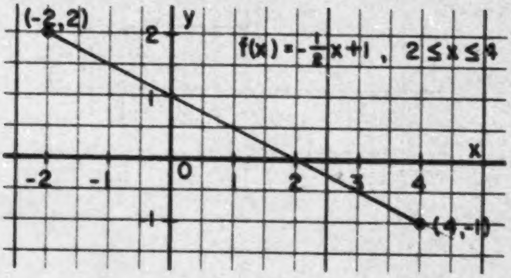
(e) Domain: all non-zero real numbers. Range: $\{-1, 1\}$.

(f) Same function as in (e).

3.



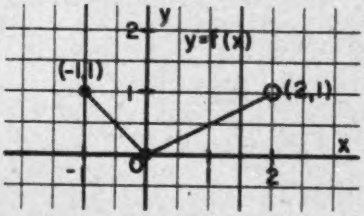
4.



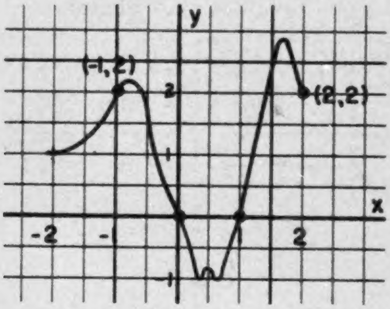
5. Domain: all x such that $-1 \leq x < 2$.

Range: all y such that $0 \leq y \leq 1$.

$$f(x) = \begin{cases} -x, & -1 \leq x \leq 0 \\ \frac{1}{2}x, & 0 < x < 2. \end{cases}$$



6. There are many functions which satisfy all of these conditions. One example is:



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For any number a in the domain of definition of a function f, there is exactly one point on the graph of the function with abscissa a, viz. (a, f(a)). Therefore the vertical line x = a will intersect the graph of f in exactly one point, viz. (a,f(a)). If a is not in the domain of definition of f, then the line x = a will not intersect the graph of f at all. The rule for f can be stated as follows: "To each real number a in the domain of definition, assign that real number b such that (a,b) is a point on the graph."

Answers to Problem Set 17-3b; pages 527-529:

1. Each of the graphs (a) - (ℓ), whether or not it is the graph of a function, determines a set consisting of all those real numbers a such that there is at least one point on the graph with abscissa equal to a. If the graph happens to be the graph of a function, then this set is the domain of definition of the function. Now, for each number a in this "domain", we associate all of those real numbers b such that (a,b) is on the graph. This association will be a function if and only if there is exactly one such number b for each a. This is the situation in cases (a), (e), (h) and (i), so that these are graphs of functions. On the other hand, in the remaining cases, there are values of a to which several different values of b are associated, so that these are not graphs of functions.

   Most of the students obviously will be unable to discuss this exercise as precisely as we have done above. However, most of them should be able to get the idea. Acceptable answers in the cases (e) and (f) might run somewhat as follows: "(e) is the graph of a function because there is only one point on the graph directly above any point on the x-axis and, therefore, the graph can be used to define an association for a function as was done in Problem 1(d) of Problem Set 17-1a.

Also, (f) is not the graph of a function because directly above some points on the x-axis there are two points on the graph, so that the graph cannot be used to define a function."

In going over these problems, try to get the students to state explicitly what the domain of definition is and how the rule might be formulated in those cases which are graphs of functions. Problems 2, 3, and 4 are intended to lead up to the general criterion for a graph to be the graph of a function.

2. (a) $h(-3) = \sqrt{3}$. This can be estimated roughly as 1.7.
$h(0) = 0$, $h(2) = -2$.

(b) The set of all x such that $-4 \leq x \leq 3$.

(c) The set of all y such that $-3 \leq y \leq 2$.

3. (a) For x in the domain of definition of g there will be one and only one point (x,y) on the graph G. The number $g(x)$ is equal to the ordinate y of the point (x,y) of the graph.

(b) The domain of definition of g is the set of all real numbers x such that there exists a point on G with abscissa equal to x.

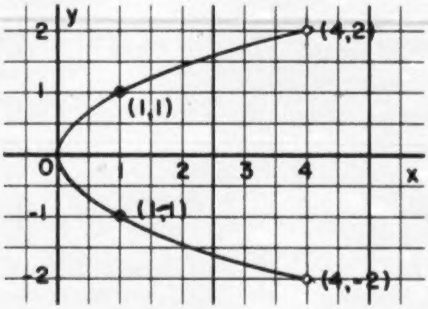
(c) To show that "if $(a,b) \neq (c,d)$ then $a \neq c$," we may prove the contrapositive of this statement, namely, "if $a = c$, then $(a,b) = (c,d)$".

Thus, if $a = c$ and if the points (a,b) and (c,d) are on the graph of g, then $b = g(a)$ and $d = g(c)$. Since g is a function, there is exactly one value $g(a)$. Hence, if $a = c$, then $g(a) = g(c)$. It follows that $b = d$, and $(a,b) = (c,d)$. This completes the proof.

4. This problem is the basis for the "ordered pair" definition of function which the student will eventually encounter if he continues in mathematics. In the "ordered pair" definition, the function is identified with its graph, i.e. a set of ordered pairs, and so the definition consists in specifying which sets of ordered pairs are wanted.

In order to show that the set of points G which satisfies the given condition is the graph of a function we need to exhibit the domain of definition and the rule for a function g so that G is the graph of g. Take the domain of definition as the set of all real numbers x such that there exists y for which the point (x,y) belongs to G. Notice that, by the condition on G, there is for each such x exactly one number y such that (x,y) belongs to G. Therefore, if we define g(x) = y for each x in the domain, the result is a function g whose graph is G.

5. This is not the graph of a function.



17-4. Linear Functions

Any line (or portion of a line) is the graph of a linear function with the exception of vertical lines. If a line is not vertical, its equation can be put into the y-form: $y = Ax + B$. Thus every linear function can be represented by a linear expression. In other words, if f is a linear function, then there exist real numbers A and B such that $f(x) = Ax + B$ for every x in the domain of definition of f.

Answers to Problem Set 17-4; pages 529-530:

1. (a) The graph is a horizontal line, $y = B$.

   (b) The graph is the x-axis.

   (c) Since the points $(-3,0)$ and $(1,2)$ are on the graph, we must have

   $$\begin{cases} 0 = A\cdot(-3) + B \\ 2 = A\cdot 1 + B \end{cases}$$

   This is a system of equations in the unknowns A and B. The solution is $A = \frac{1}{2}$ and $B = \frac{3}{2}$. The problem can also be solved by obtaining the equation of the line determined by the points $(-3,0)$, $(1,2)$ and then writing it in the y-form.

   (d) The domain of definition is the set of all real numbers x such that $-3 \le x \le 1$.

   (e) $$\begin{cases} 1 = A\cdot(-1) + B \\ 3 = A\cdot 3 + B \end{cases}$$

   Solving for A and B, we obtain $A = \frac{1}{2}$ and $B = \frac{3}{2}$. Again, we could obtain A and B by writing the equation of the line determined by the points $(-1,1)$, $(3,3)$ in the y-form. Note that this is the same line but a different function from that in (c).

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(f) The slope is $\frac{1}{2}$ and the y-intercept number is $\frac{3}{2}$.

(g) The domain of definition is the set of all real x such that $-1 < x < 3$.

2. The equation of the line L is $x + 2y + 1 = 0$. When $y = 2$, $x = -5$ and, when $y = -2$, $x = 3$. Therefore

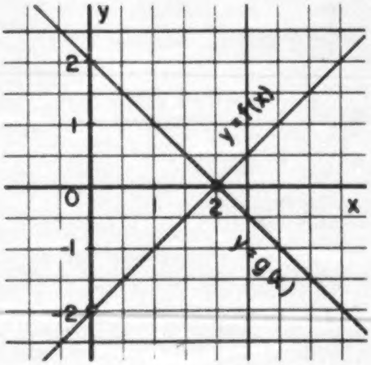
$$h(x) = -\frac{1}{2}x - \frac{1}{2}, \quad -5 < x < 3.$$

3. (a) linear (d) not linear
(b) not linear (e) linear
(c) not linear (f) not linear

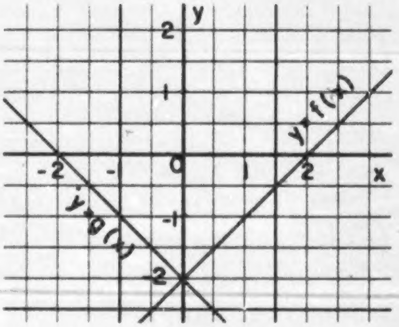
4. (a) $g(x) = -f(x)$ (d) $g(x) = f(|x|)$
(b) $g(x) = |f(x)|$ (e) $g(x) = f(-x)$
(c) $g(x) = \frac{1}{f(x)}$ (f) $g(x) = f(x^2)$

5. (a) The graph of g is obtained by rotating the graph of f one-half revolution about the x-axis.

(a)



(e)



(e) The graph of g is obtained by rotating the graph of f one-half revolution about the y-axis.

6. The graph of the equation $\left(y - F(x)\right)\left(y - G(x)\right) = 0$ is the set of all points on either the graph of F or the graph of G.

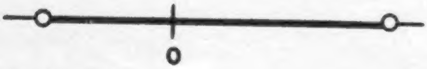
Answers to Problem Set 17-5a; Pages 532-534:

1. (a) $f(-2) = -11$; $f(-\frac{1}{2}) = -19\frac{1}{4}$; $f(0) = -21$; $f(\frac{3}{4}) = -22\frac{11}{16}$; $f(3) = -21$; $f(a) = a^2 - 3a - 21$; $f(\frac{a}{2}) = \frac{a^2}{4} - \frac{3a}{2} - 21$;
$f(a + 1) = (a + 1)^2 - 3(a + 1) - 21 = a^2 - a - 23$.

   (b) $g(-2) = 10$; $g(-\frac{1}{2}) = -1\frac{1}{4}$; $g(0) = -2$; $g(3)$ is undefined. (Note that the domain of $g(x)$ is $-3 < x < 3$.)
$g(2t - 1) = 3(2t - 1)^2 - 2 = 12t^2 - 12t + 1$ for $-1 < t < 2$. (Note: $-3 < x < 3$, hence $-3 < 2t - 1 < 3$ or $-1 < t < 2$.)

   (c)
$$f(x) = x^2 - 3x - 21 = (x - \tfrac{3}{2})^2 - \tfrac{93}{4}$$
$$= \left((x - \tfrac{3}{2}) + \tfrac{\sqrt{93}}{2}\right)\left((x - \tfrac{3}{2}) - \tfrac{\sqrt{93}}{2}\right)$$
$$= \left(x - (\tfrac{3}{2} - \tfrac{\sqrt{93}}{2})\right)\left(x - (\tfrac{3}{2} + \tfrac{\sqrt{93}}{2})\right) = 0$$
Hence, $\left\{\frac{3 + \sqrt{93}}{2}, \frac{3 - \sqrt{93}}{2}\right\}$ is the truth set.

   (d)



$\frac{3 - \sqrt{93}}{2} \approx -3.3$ $\frac{3 + \sqrt{93}}{2} \approx 6.3$

   (e) $f(t) + g(t) = 4t^2 - 3t - 23$. Note that if one function is defined for all real numbers and the other function is defined for $-3 < t < 3$, then the sum of the two functions is defined for $-3 < t < 3$.

(f) $f(a) + 3 = a^2 - 3a - 18$; $f(a + 3) = (a + 3)^2 - 3(a + 3) - 21 = a^2 + 3a - 21$; $3f(a) = 3a^2 - 9a - 63$;

$f(3a) = (3a)^2 - 3(3a) - 21 = 9a^2 - 9a - 21$.

(g) All in part (f) are quadratic polynomials.

(h) $f(t)g(t) = (t^2 - 3t - 21)(3t^2 - 2)$

$= 3t^4 - 9t^3 - 65t^2 + 6t + 42$

Note that as in case (e) the product of the two functions is defined for $-3 < t < 3$.

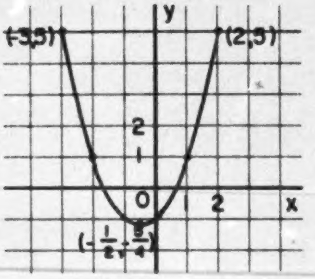
(i) The result in (e) is a quadratic polynomial and in (h) the polynomial is not quadratic.

2. (a) $A = \frac{1}{2}b(b + 10)$ is a quadratic polynomial; domain (of definition): every positive real number.

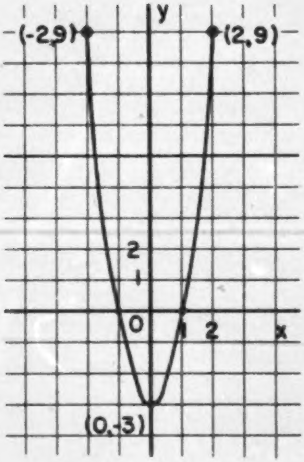
(b) If the smaller number is denoted by s, then the larger number is $120-2s$, and the product $P = s(120 - 2s)$ is a quadratic polynomial in s. Since both numbers are positive, the domain is the set of real numbers s such that $0 < s < 40$. Note that this restriction of the domain is necessary because of the condition that s is the smaller of the two numbers.

(c) If L is the length of the side parallel to the wall, then the length of the side perpendicular to the wall is $\frac{1}{2}(120 - L)$, and the area A of the rectangular pen is: $A = \frac{1}{2}L(120 - L)$. The area is a quadratic polynomial in L; the domain (of definition) is the set of real numbers L such that $0 < L < 120$.

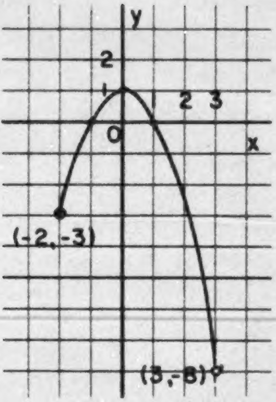
3. (a)



(b)



(c)



4. We know that the product of two positive or two negative real numbers is positive. Since $x^2 = x \cdot x$, it follows that x^2 is a product of two positive numbers if $x > 0$, or of two negative numbers, if $x < 0$, that is $x^2 > 0$ for any real number $x \neq 0$. We know further that if $ab = 0$, where a and b are real numbers, then at least one of them is zero. Since $x^2 = x \cdot x = 0$, it follows that $x = 0$. Conversely, if $x = 0$, then $x^2 = x \cdot x = 0 \cdot 0 = 0$. Since points with positive ordinates are above the x-axis, it follows that the graph of $y = x^2$ has positive ordinates for all $x \neq 0$ and a single point $(0,0)$, for $x = 0$, lies on the x-axis.

5. Note that by the graph of the function $y = x^2$ we mean the graph of the open sentence $y = x^2$. If (a,b) is a point on the graph, then $b = a^2$ is true. Since $b = (-a)^2 = a^2$, it follows that the open sentence is also true for the ordered pair $(-a,b)$; in other words, $(-a,b)$ is also on the graph.

6. If x is positive, multiplication of the members of "$x < 1$" by x yields $x^2 < x$. If, for the same value of x, the ordinate of $y = x^2$ is denoted by y_1 and the ordinate of $y = x$ by y_2, then for $0 < x \le 1$ we have $y_1 < y_2$. In other words, the graph of $y = x^2$ lies below the graph of $y = x$.

7. Here, as in Problem 6, we obtain $x < x^2$ and $y_2 < y_1$. Hence, for $x > 1$, the graph of $y = x^2$ lies above the line $y = x$.

8. Multiplication of the members of "$a < b$" by a and b yields $a^2 < ab$ and $ab < b^2$, respectively. Hence, by the transitive property of order we obtain $a^2 < b^2$.
If we denote $y_a = a^2$ and $y_b = b^2$ then by the above property for $b > a$ we obtain $y_b > y_a$ for all $b > a > 0$. Hence it follows that the graph of $y = x^2$ rises steadily as we move from 0 to the right.

9. The horizontal line $y = a$ where $a \geq 0$ (since the graph of $y = x^2$ is above the x-axis, a cannot be negative) and the graph of $y = x^2$ have equal ordinates at the points of intersection. Therefore, $x^2 = a$.
Since $x^2 - a = (x - \sqrt{a})(x + \sqrt{a})$, $x^2 - a = 0$ has the truth set $\{\sqrt{a}, -\sqrt{a}\}$ for $a \neq 0$ and the truth set $\{0\}$ for $a = 0$, it follows that there can be at most two points of intersection.

10. Since the slope of a line containing the points (a,b) and (c,d) is $\frac{d - b}{c - a}, (c \neq a)$, we obtain easily for the points $(0,0)$ and (a,a^2) the slope $\frac{a^2 - 0}{a - 0} = a$. Hence, we conclude that the slope of the line containing $(0,0)$ and (a,a^2) approaches 0 as a approaches 0. But a line passing through $(0,0)$ with a slope close to zero apparently

approaches the x-axis, which touches the parabola at $(0,0)$. If we note that the segment of the line between $(0,0)$ and (a,a^2), for a close to 0, nearly coincides with the arc of the parabola, it is plausible that the graph must be flat near the origin.

Answers to Problem Set 17-5b: pages 535-536:

1. (a) The smaller the value of a, the flatter the graph of $y = ax^2$, and for $a = 0$ it degenerates into the x-axis. All parabolas for $0 < a < 1$ are between the x-axis and the parabola $y = x^2$.

   (b) For $a > 1$ the parabolas $y = ax^2$ are inside the parabola $y = x^2$.

   (c) The graph of $y = ax^2$ is between the x-axis and the parabola $y = -x^2$ if $-1 < a < 0$.

   (d) The graph of $y = ax^2$ is inside the parabola $y = -x^2$ if $a < -1$.

   (e) The graph of $y = ax^2$ is very close to the y-axis for $|a|$ very large.

2. The graphs of $y = x^2 + k$ and $y = x^2$ differ only in location. The graph $y = x^2 + k$ lies $|k|$ units upward if $k > 0$ and $|k|$ units downward if $k < 0$.

3. $y = (x - h)^2$ and $y = x^2$ differ only in location; namely, $y = (x - h)^2$ is $|h|$ units to the right of $y = x^2$ if $h > 0$, and $|h|$ units to the left if $h < 0$.

4. (a) $y = (x + 1)^2$ has exactly the shape of $y = x^2$, and is located one unit to left of it.

   (b) $y = -3x^2$ is inside the parabola $y = -x^2$, closer to the y-axis.

(c) $y = x^2 - 3$ has exactly the shape of the parabola $y = x^2$, and is located 3 units downward.

(d) $y = -(x - 1)^2$ has exactly the shape of the parabola $y = -x^2$ and is one unit to the right of it.

(e) $y = 2(x - 2)^2$ has exactly the shape of $y = 2x^2$ which is inside of $y = x^2$, and is 2 units to the right of $y = 2x^2$.

(f) $y = (x + 1)^2 + 1$ has exactly the shape of the parabola $y = x^2$, and is located one unit to the left and one unit upward from $y = x^2$.

(g) $y = 2(x - 1)^2 - 1$ has exactly the shape of $y = 2x^2$, and is one unit to the right and one unit downward from $y = 2x^2$.

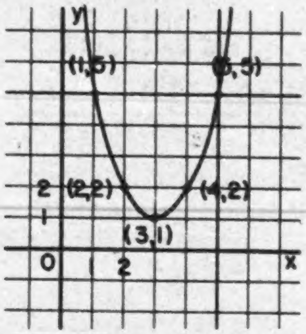
(h) $y = -2(x + 1)^2 - 1$ has exactly the shape of $y = -2x^2$ which is inside $y = -x^2$ and located one unit to the left and one unit downward from the parabola $y = -2x^2$.

5. The graph $y = a(x - h)^2 + k$ can be obtained by moving the graph $y = ax^2$ in the positive direction of the **x-axis** $|h|$ units, if $h > 0$, and in the negative direction of the x-axis if $h < 0$, and $|k|$ units upward if $k > 0$, and $|k|$ units downward, if $k < 0$. The vertex and the equation of the axis of the parabola are (h, k) and $x = h$, respectively.

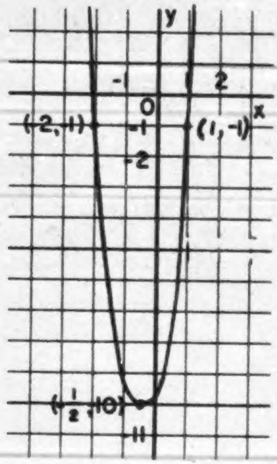
6. $x = a(y - 1)^2 - 1$. Since a is arbitrary, there are infinitely many such parabolas.

Answers to Problem Set 17-6; pages 538-539:

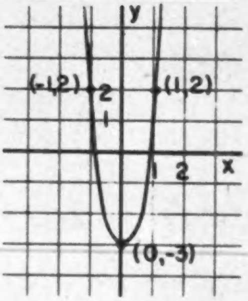
1. (a) $y = x^2 - 6x + 10 = (x - 3)^2 + 1$



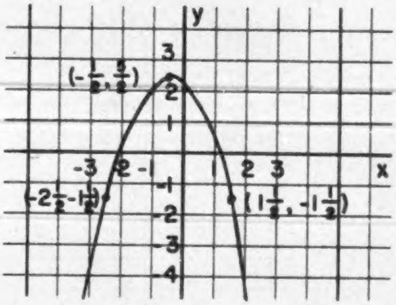
(b) $y = 4x^2 + 4x - 9 = 4(x + \frac{1}{2})^2 - 10$



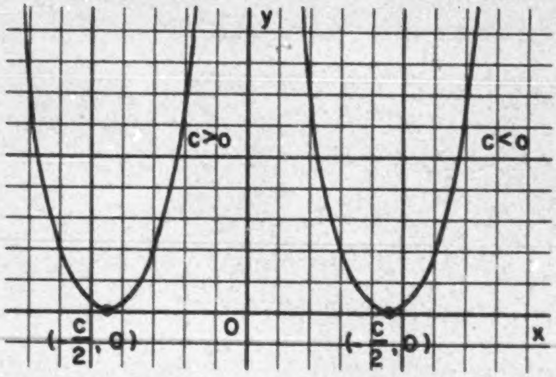
(c) $y = 5x^2 - 3$



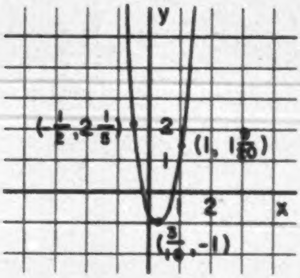
(d) $y = -x^2 - x + \frac{9}{4} = -(x + \frac{1}{2})^2 + \frac{5}{2}$



(e) $y = 4x^2 + 4cx + c^2 = (2x + c)^2 = 4(x + \frac{c}{2})^2$



(f) $y = 5x^2 - 3x - \frac{11}{20} = 5(x^2 - \frac{3}{5}x + \frac{9}{100}) - \frac{11}{20} - \frac{9}{20}$
$= 5(x - \frac{3}{10})^2 - 1$



2. (a) no points.

(b) $(\frac{-1 + \sqrt{10}}{2}, 0)$ and $(\frac{-1 - \sqrt{10}}{2}, 0)$

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2. (c) $(\sqrt{\frac{3}{5}},0)$ and $(-\sqrt{\frac{3}{5}},0)$

(d) $(\frac{-1+\sqrt{10}}{2},0)$ and $(\frac{-1-\sqrt{10}}{2},0)$

(e) $(-\frac{c}{2},0)$

(f) $(\frac{3+2\sqrt{5}}{10},0)$ and $(\frac{3-2\sqrt{5}}{10},0)$

3. $$Ax^2 + Bx + C = A(x^2 + \frac{B}{A}x + \frac{B^2}{4A^2}) + C - \frac{B^2}{4A}, \quad A \neq 0$$

$$= A(x + \frac{B}{2A})^2 + \frac{4AC - B^2}{4A}$$

$$= a(x - b)^2 + k,$$

where $a = A$, $h = -\frac{B}{2A}$, $k = \frac{4AC - B^2}{4A}$

*4. (a) $a(x - h)^2 + k = ax^2 - 2ahx + (ah^2 + k) = 3x^2 - 7x + 5$

If $a = 3$, $-2ah = -7$, $ah^2 + k = 5$, then

$a = 3$ implies that $-2(3)h = -7$; i.e. $h = \frac{7}{6}$;

$a = 3$ and $h = \frac{7}{6}$ implies that $(3)(\frac{7}{6})^2 + k = 5$, i.e. $k = \frac{11}{12}$.

Hence, $3x^2 - 7x + 5 = 3(x - \frac{7}{6})^2 + \frac{11}{12}$.

(b) $ax^2 - 2ahx + (ah^2 + k) = 5x^2 - 3x + \frac{13}{20}$

If $a = 5$, $-2ah = -3$, $ah^2 + k = \frac{13}{20}$, then

$a = 5$, $h = \frac{3}{10}$, $h = \frac{1}{5}$.

Hence, $5x^2 - 3x + \frac{13}{20} = 5(x - \frac{3}{10})^2 + \frac{1}{5}$

(c) $ax^2 - 2ahx + (ah^2 + k) = Ax^2 + Bx + C$, for every real number x. This is possible if

$a = A$, $-2ah = B$, and $ah^2 + k = C$.

If $a = A$, then the sentence "$-2ah = B$" is equivalent to "$-2Ah = B$," that is, $h = \frac{B}{2A}$. Also, if $a = A$ and $h = -\frac{B}{2A}$, then "$ah^2 + k = C$" is equivalent to "$A \cdot \frac{B^2}{4A^2} + k = C$," that is, $k = C - \frac{B^2}{4A} = \frac{4AC - B^2}{4A}$.

Answers to Problem Set 17-7; pages 544-545:

1. (a) not factorable

(b) $6x^2 - x - 12 = (3x + 4)(2x - 3)$

In this case, completion of the square would result in the same factors, but we should first try to find factors over the integers by the methods of Chapter 12.

(c) $\frac{1}{2}x^2 + 4x + 6 = \frac{1}{2}(x^2 + 8x + 12)$

$= \frac{1}{2}(x + 6)(x + 2)$

(d) $4y^2 + 2y + \frac{1}{4} = (2y + \frac{1}{2})^2$

(e) not factorable

(f) $2 - 2z - z^2 = 3 - (z + 1)^2 = (\sqrt{3} + z + 1)(\sqrt{3} - z - 1)$.

(g) $1 - 5x^2 = (1 - \sqrt{5}x)(1 + \sqrt{5}x)$.

(h) not factorable.

(i) $5v^2 - 5v - \frac{11}{4} = 5(v - \frac{1}{2})^2 - 4$

$= (\sqrt{5}(v - \frac{1}{2}) + 2)(\sqrt{5}(v - \frac{1}{2}) - 2)$.

(j) $x^2 + (a + b)x + ab = (x + \frac{a+b}{2})^2 - (\frac{a-b}{2})^2 = (x + a)(x + b)$.

2. (a) $\left\{\frac{2}{\sqrt{3}}, \frac{-2}{\sqrt{3}}\right\}$

(b) $4 - x - 3x^2 = (4 + 3x)(1 - x) = 0.$ $\{-\frac{4}{3}, 1\}.$

(c) $\emptyset.$

(d) $s^2 - s - \frac{1}{2} = (s - \frac{1}{2})^2 - \frac{3}{4} = (s - \frac{1}{2} - \frac{\sqrt{3}}{2})(s - \frac{1}{2} + \frac{\sqrt{3}}{2}) = 0.$

$\{\frac{1 + \sqrt{3}}{2}, \frac{1 - \sqrt{3}}{2}\}$

(e) $\frac{4}{5}t^2 + \frac{4}{5}t + \frac{1}{5} = \frac{1}{5}(4t^2 + 4t + 1) = \frac{1}{5}(2t + 1)^2 = 0.$ $\{-\frac{1}{2}\}$

(f) $\frac{1}{3}y^2 + 2y - 3 = \frac{1}{3}(y + 3)^2 - 6 = \frac{1}{3}\left((y + 3)^2 - 18\right)$

$= \frac{1}{3}(y + 3 + \sqrt{18})(y + 3 - \sqrt{18}) = 0.$

$\{-3 - 3\sqrt{2}, -3 + 3\sqrt{2}\}.$

(g) $\emptyset.$

(h) $3n^2 - 7n = n(3n - 7) = 0.$ $\{0, \frac{7}{3}\}.$

3. (a) From the fact that

$a(x - h)^2 + k = a\left((x - h)^2 + \frac{k}{a}\right), \quad a \neq 0$

it follows that it is factorable if $\frac{k}{a} \leq 0.$

(b) If $\frac{k}{a} = -p^2$ where $p = \frac{m}{n}$, m and n are integers, then $a(x - h)^2 + k = a\left((x - h)^2 - p^2\right) = a(x - h - p)(x - h + p)$ $= \frac{a}{n^2}\left(n(x - h) - m\right)\left(n(x - h) + m\right)$ where m, n, h are integers with a possible exception of the constant factor $\frac{a}{n^2}$. If $n = 1$, that is, p an integer, then $\frac{a}{n^2}$ is also an integer. Hence, if $-\frac{k}{a}$ is a perfect square of an integer, the above polynomial is perfectly factorable over the integers.

(c) If $\frac{k}{a} < 0$, $\frac{k}{a} = 0$, or $\frac{k}{a} > 0$ the truth set of $a(x - h)^2 + k$ contains two, one, or no real numbers, respectively.

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4. (a) If the length of its longer side is x inches, then the length of the shorter side is $(6 - x)$ inches. Thus

$x(6 - x) = 7$ or $x^2 - 6x + 7 = 0$

$x^2 - 6x + 7 = (x-3)^2 - 2 = (x - 3 -\sqrt{2})(x - 3 +\sqrt{2}) = 0.$

Hence, $x = 3 \pm \sqrt{2}$. The rectangle is $(3+\sqrt{2})$ inches wide.

(b) If one side is x inches, then the second side is $(x - 1)$ inches and the hypotenuse is $x + 2$. By the Pythagorean Theorem we have

$x^2 + (x - 1)^2 = (x + 2)^2$ or $x^2 - 6x - 3 = 0$

$x^2 - 6x - 3 = (x - 3)^2 - 12$

$= (x - 3 -\sqrt{12})(x - 3 +\sqrt{12}) = 0.$

Hence $x = 3 \pm \sqrt{12}$. The required side is $3+2\sqrt{3}$ inches long.

(c) If one number is x, then the second is $5 - x$, and the product $x(5 - x) = 9$, or $x^2 - 5x + 9 = 0$. Since the truth set of $x^2 - 5x + 9 = (x - \frac{5}{2})^2 + \frac{11}{4} = 0$ contains no real numbers, the problem has no solution.

5. In Problem 3 in Problem Set 17 - 6 we obtained the relation

(a) $Ax^2 + Bx + C = A(x + \frac{B}{2A})^2 - \frac{B^2-4AC}{4A}$

$= A(x + \frac{B}{2A})^2 - \frac{B^2 - 4AC}{4A^2}$

(b) If $B^2 - 4AC < 0$, then $(x + \frac{B}{2A})^2 - \frac{B^2 - 4AC}{4A^2} > 0$;
The parabola $y = Ax^2 + Bx + C$ is above or below the x-axis (above for $A > 0$ and below for $A < 0$), that is, $Ax^2 + Bx + C = 0$ has no solution.

(c) If $B^2 - 4AC = 0$, then $y = Ax^2 + Bx + C = A(x + \frac{B}{2A})^2$, and the parabola is above or below the x-axis and touches the x-axis at $(-\frac{B}{2A}, 0)$; that is, $Ax^2 + Bx + C + 0$ has one solution $x = \frac{-B}{2A}$.

(d) If $B^2 - 4AC > 0$, then $A\left(\left(x + \frac{B}{2A}\right)^2 - \frac{B^2 - 4AC}{4A^2}\right) =$

$$A\left(x + \frac{B}{2A} + \frac{\sqrt{B^2 - 4AC}}{2A}\right)\left(x + \frac{B}{2A} - \frac{\sqrt{B^2 - 4AC}}{2A}\right) = 0$$

has the solution set

$$\left\{\frac{-B + \sqrt{B^2 - 4AC}}{2A}, \frac{-B - \sqrt{B^2 - 4AC}}{2A}\right\}.$$

This is the familiar quadratic formula which is often stated without proof. It is given here only as a generalization of the technique of factoring a quadratic polynomial by completing the square. Do *not* ask students to memorize a formula. The important point is that *every* quadratic polynomial can be factored, and, therefore, *every* quadratic equation can be solved.

Chapter 17

Suggested Test Items

1. In each of the following describe (if possible) the function (i) by a table, (ii) by an expression in x. In each case describe the domain of definition and the range.

   (a) To each positive real number assign the sum of 2 and twice the number.

   (b) Associate with each integer the reciprocal of the integer.

(c) To each real number assign the ordinate of the point on the line with slope 2 and y-intercept number -2 whose abscissa is the number.

2. What is the domain of the function defined by the expression $\sqrt{2x - 4}$? What is the range of this function?

3. Given the function g defined as follows:

$$g(x) = x - \frac{1}{x}, \text{ for each non-zero real number } x.$$

What real numbers are represented by

(a) $g(-2)$ (e) $|g(-\frac{1}{2})|$

(b) $g(-\frac{1}{2})$ (f) $\frac{1}{2}g(-1)$

(c) $-g(\frac{1}{2})$ (g) $g(\frac{a}{2})$, $a > 0$

(d) $g(|-\frac{1}{2}|)$ (h) $-g(-a)$, $a > 0$

4. Consider the function F defined by

$$F(x) = \begin{cases} 2, & -2 \leq x < 0 \\ x+2, & 0 < x \leq 2 \end{cases}$$

(a) What is the domain of definition of F?

(b) What numbers are represented by $F(-\frac{3}{2})$, $F(\frac{1}{2})$, $F(0)$, $F(-\frac{2}{3})$, $F(\frac{5}{2})$?

(c) Draw the graph of F.

(d) What is the truth set of the sentence $F(x) = 3$?

5. Give a rule for the definition of the function whose graph is the line extending from (-1,2) to (4,1), including the endpoints.

6. Is every line the graph of a function? If not, give some exceptions.

7. If f is the linear function defined by

   $f(x) = mx + b$, for all real numbers x,

   (a) describe the graph of f if $b = 0$ and $m > 0$,

   (b) describe the graph of f if $m = 0$ and $b > 0$,

   (c) determine m and b if the graph of f contains the points $(-2,3)$ and $(3,-2)$.

8. Let F and G be defined by:

   $F(x) = 3x^2 + 2x - 4, \quad -2 < x < 2,$

   $G(x) = -x^2 + 2 \quad , \quad -1 < x < 3.$

   (a) Determine $F(2) - G(1)$, $F(-4) + G(-4)$.

   (b) Determine $F(x) + G(x)$, $-1 < x < 2$.

   (c) Determine $F(x)\,G(x)$, $-1 < x < 2$.

   (d) With respect to the same set of axes, draw the graphs of F and G.

   (e) What is the truth set of the sentence "$F(x) = G(x)$", $-1 < x < 2$?

   (f) Write $F(x)$ in standard form and find the vertex and the axis of the graph of F.

   (g) Solve the quadratic equation $F(x) = 0$, $-2 < x < 2$.

9. If a quadratic equation is written in the form

   $a(x - b)^2 + k = 0,$

   discuss the values of h and k for which the equation has

   (a) two real solutions,

   (b) one real solution,

   (c) no real solution.

10. The first side of a rectangle is x inches in length; its other side is 2 inches shorter, and the diagonal is 3 inches longer than the first side. What is the value of x?
